TESTING UNIFORMITY ON THE HYPERSPHERE

By

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To my parents

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A well known test for uniformity on the circle given by B. Ajne has been extended to the hypersphere by numerous authors. However, under the hypothesis of uniformity the asymptotic distribution of Ajne's test statistic on the hypersphere is somewhat difficult to work with since it involves an infinite series of independent, weighted, central, chi-square random variables. Presented here is a new test statistic for testing uniformity on the hypersphere which has a normal asymptotic distribution under both the null hypothesis of uniformity and rotationally symmetric, unimodal alternatives.

Relations of the new test statistic to Ajne's generalized test statistic and Rayleigh's test statistic are given. The new test is shown to have certain optimal properties against Langevin, generalized cardioid and other selected alternatives.

In order to discuss rotationally symmetric, unimodal densities on the hypersphere, representations in terms of Gegenbauer polynomials are considered. The coefficients for these representations, called

Gegenbauer transforms, are given for many well known densities. Also, for densities represented by means of Gegenbauer transforms, a new graphical curve fitting aid is presented.

In the course of investigating Gegenbauer polynomials, a standardization proposed by J. G. Saw for the Gegenbauer polynomials and a weight function differing by a constant factor from the usual weight function for Gegenbauer polynomials are shown to simplify certain results. Many formulas for Gegenbauer polynomials are collected and written in terms of Saw's standardization and this modified weight function.

CHAPTER 1
INTRODUCTION AND PRELIMINARIES

1.1 Introduction

Classical multivariate statistical methods deal with data assuming values in E^m, Euclidean space of m dimensions consisting of $m \times 1$ vectors with real components. For our purposes, we consider data restricted to S_m, the surface of the unit m-sphere given by

$$S_m = \{\underline{x} : \underline{x} \in E^m, \underline{x}'\underline{x} = 1\}.$$

The case $m = 1$ is degenerate in as much as $S_1 = \{-1,1\}$ so that $x \in S_1$ is a Bernoulli random variable. For $m > 2$, S_m is the domain of what are commonly called "directional data" since a vector $\underline{x} = (x_1,\ldots,x_m)' \in S_m$ may be expressed in terms of the "directions" or polar angles $\theta_1,\theta_2,\ldots,\theta_{m-1}$ by

$$x_i = \begin{cases} \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{m-1} & \text{if } i = 1, \\ \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{m-i} \cos\theta_{m-i+1} & \text{if } 2 \leq i \leq m-1, \\ \cos\theta_1 & \text{if } i = m, \end{cases}$$

where to assure a one-to-one transformation we require $0 < \theta_j \leq \pi$, $1 \leq j \leq m-2$ and $0 < \theta_{m-1} < 2\pi$.

Most instances in which directional data arise involve observing the motion of some experimental object. Consequently, much attention has historically been given to the circular case, $m = 2$, and the spherical case, $m = 3$, while relatively little work has been devoted to higher dimensional cases, $m > 3$. For an account of much of the current statistical methodology surrounding directional data on the circle (2-sphere) and sphere (3-sphere) one may refer to Mardia (1972).

A statistical methodology dichotomized according to directional data on the circle and sphere is undesirable for two reasons. First, although comparably rare at present, there exist applications of directional data on m-spheres with $m > 3$ as indicated in section 1.2. Second, there is a certain redundancy in the literature in which methods now treated separately for the circle and sphere may be unified under a more general framework.

For example, the von Mises distribution on S_2 and the Fisher distribution on S_3 are treated separately in the literature. However, both are special cases of a general distribution on S_m introduced by Watson and Williams (1956) and now called the Langevin distribution by Watson (1983). Moreover, there are tests in the literature prescribed for $m = 2$ and tests prescribed for $m = 3$ which turn out to be exact analogues; the Rayleigh test is one example. Elsewhere there are test prescriptions for the circle for which there are no immediately apparent counterparts for the sphere; the range test is one instance. It is revealing that Mardia's text treats the circular and spherical cases separately. While readily admitting that most applications occur for $m = 2$ or $m = 3$, a desire to study these two cases simultaneously and unify certain methods, necessitates the consideration of procedures applicable to "general m," $m \geq 2$.

1.2 Occurrence of Directional Data

Examples of directional data arise from a wide variety of sources. In astronomy, directional methods are used to study the motion of planets and comets; the orbital planes of our sun's planets have been the subject of statistical analysis from Daniel Bernoulli (1734) to G. S. Watson (1970). Examples in biology include bird migration, swimming direction of daphnia in polarized light and orientation of turtles after treatment. In geography, directional data are often recorded in terms of longitude and latitude, for instance, the coordinates of earthquake epicenters. Directional data are used in geology to understand remanent magnetism (pole wandering), tectonic forces (such as continental drift), lava flow patterns and orientation of cross bedding for interpreting directions of depositing currents of wind or water. Of interest in medicine is vector cardiographic data, monthly onset of a disease and monthly death rates. Meteorological studies employ directional data to describe variables such as wind directions, monthly rainfall and monthly evaporation from a reservoir. In physics, directional data have been used to study bubble chamber data, atomic weights and Brownian motion. The perception of direction is studied in physiology and psychology. Additionally, circular distributions are used to describe such diverse topics as certain periodic economic data, roulette wheels, rounding errors in numerical calculations and the generation of random numbers (due to the periodicity of remainders). For a discussion of the examples above, or for actual data sets, one may refer to Batschelet (1965), Mardia (1972) and Watson (1983).

On occasion we find directional data problems which require more than three dimensions for analysis. For instance, P. J. Laycock (see

Mardia (1975)) cites multivariate data taken on fifteen orthodontic planes on human skulls. A linear model indexed by a "direction" in fifteen dimensions was then examined. Further, Stephens (1982) proposed an analysis of variance technique employing the von Mises distribution to analyze a set of m proportions of a continuum such as time or volume. Stephens' procedure requires analysis on S_m for $m \geq 2$.

1.3 The Uniform Distribution on the Hypersphere

Of central importance in developing directional data methods for use on general hyperspheres S_m, $m \geq 2$, is testing whether or not the data exhibit some "preferred direction." In particular, the uniform distribution on S_m provides the null hypothesis of no preferred direction. In section 1.5 we will more fully describe the role of the uniform distribution in directional data and in section 1.6 we will survey methods used to test uniformity on S_m.

In order to define the uniform distribution and other commonly used distributions on S_m and, indeed, in order to carry out integration on S_m, we first need to examine the topology of S_m.

We begin by defining the "cap,"

$$C_m(\underline{\xi}, u) = \{ \underline{x} : \underline{x} \in S_m, \underline{x}'\underline{\xi} > u \},$$

to be an open set on S_m. In other words, a set $G \subset S_m$ is open if and only if for each $\underline{x} \in G$, there exist $\underline{\xi} = \underline{\xi}(\underline{x})$ and $u = u(\underline{x})$ such that $\underline{x} \in C_m(\underline{\xi}(\underline{x}), u(\underline{x})) \subset G$. Now let A_m be the collection of all "caps," that is,

$$A_m = \{ C_m(\underline{\xi}, u) : \underline{\xi} \in S_m, |u| < 1 \}.$$

and define B_m to be the σ-algebra generated by A_m.

To complete the formation of a probability space on S_m, a suitable measure must be defined. Although any non-negative measure, μ, on the σ-algebra B_m with $\mu(S_m) = 1$ will suffice, we adopt an invariant measure $U^{(\nu)}$, $\nu = m/2 - 1$, in which $U^{(\nu)}(A) = U^{(\nu)}(PA)$ for P an orthogonal matrix and $A \in B_m$. Moreover, we will see that our choice of measure $U^{(\nu)}$ conforms to the usual notion of arc length on the circle S_2 and surface area of spherical caps on the sphere S_3.

Specifically, let $U^{(\nu)}$ be a non-negative measure defined on the σ-algebra B_m of subsets of S_m with

$$U^{(\nu)}(C_m(\underline{\xi},u)) = \begin{cases} 1 & \text{if } -\infty < u < -1, \\ \displaystyle\int_u^1 \frac{(1-x^2)^{\nu-1/2}}{B(1/2,\ \nu+1/2)}\,dx & \text{if } -1 \leq u \leq 1, \\ 0 & \text{if } 1 < u < \infty, \end{cases}$$

where $B(a,b)$ is the beta function with arguments a and b. Note that $U^{(\nu)}(S_m) = 1$ and that the invariance property of $U^{(\nu)}$ requires $P(\underline{x} \in C_m(\underline{\xi},u))$ to be independent of $\underline{\xi}$. For future reference we may also write

$$U^{(\nu)}(C_m(\underline{\xi},u)) = 1 - \int_{-1}^{u} \frac{(1-x^2)^{\nu-1/2}}{B(1/2,\ \nu+1/2)}\,dx$$

$$= 1 - W^{(\nu)}(u), \text{ say.}$$

To see the connection between arc length in the circular case, $m = 2$ ($\nu = 0$), and the measure $U^{(0)}$, observe that the arc length of $C_2(\underline{\xi},u)$ is $2\cos^{-1}u$. Correspondingly, the probability measure $U^{(0)}$ gives

$$U^{(0)}(C_2(\underline{\xi},u)) = \int_u^1 \frac{(1-x^2)^{-1/2}}{B(1/2,\ 1/2)}\,dx$$

which, upon setting $x = \cos\theta$,

$$= \int_0^{\cos^{-1}u} \frac{1}{\pi}\,d\theta$$

$$= \frac{2\cos^{-1}u}{2\pi}$$

$$= \frac{\text{arc length of } C_2(\underline{\xi},u)}{\text{circumference of } S_2}.$$

For the spherical case, $m = 3$ ($\nu = 1/2$), the surface area of the spherical cap $C_3(\underline{\xi},u)$ is $2\pi(1 - u)$. The probability measure $U^{(1/2)}$ yields

$$U^{(1/2)}(C_3(\underline{\xi},u)) = \int_u^1 \frac{1}{B(1/2,\ 1)}\,dx$$

$$= \frac{2\pi(1-u)}{4\pi}$$

$$= \frac{\text{surface area of } C_3(\underline{\xi},u)}{\text{surface area of } S_3}.$$

With the probability space $(S_m, B_m, U^{(\nu)})$ now at our disposal, we may define a density $f(\underline{u})$ of S_m with respect to $U^{(\nu)}$. Let $\underline{u} \in S_m$, then $f(\underline{u})$ necessarily satisfies:

(1) $f(\underline{u}) > 0$ almost everywhere (a.e.), that is, for every $\underline{u} \in S_m$ except on a set of zero $U^{(\nu)}$ measure;

(2) $\int_{S_m} f(\underline{u})\, dU^{(\nu)}(\underline{u}) = 1$.

Now let $\underline{u}$ be a vector valued random variable taking values on S_m with constant density $f(\underline{u}) = 1$. Then we shall say that $\underline{u}$ is uniformly distributed on S_m with respect to the probability measure $U^{(\nu)}$, denoted $\underline{u} \sim U(S_m)$. It should be noted that the invariant measure $i_m(\cdot) = \alpha_m U^{(\nu)}(\cdot)$ is frequently used in the literature. Here, $\alpha_m = \int_{\underline{x}'\underline{x}=1} 1 d\underline{x} = 2\pi^{m/2}/\Gamma(m/2)$ is the surface area of the unit m-sphere. In this case, $i_m(S_m) = \alpha_m$ so that the uniform density with respect to the measure i_m becomes the constant $1/\alpha_m$. For the most part, we shall adhere to the notationally simpler probability measure $U^{(\nu)}$, $\nu = m/2 - 1$.

We now present a useful theorem regarding the uniform distribution on S_m.

Theorem 1.3.1: Let $\underline{z}$ have an m-variate normal distribution with mean vector $\underline{0}$ and covariance matrix I_m (denoted $\underline{z} \sim N_m(\underline{0}, I_m)$), and put $r = \|\underline{z}\| = (\underline{z}'\underline{z})^{1/2}$ and $\underline{u} = \underline{z}/r$. Then r^2 has a chi-square distribution with m degrees of freedom (denoted $r^2 \sim \chi^2_m$), and $\underline{u} \sim U(S_m)$ independently of r^2.

Proof: From Saw (1970, 1973) we find the Jacobian of the transformation from $\underline{z}$ to $(\underline{u}, r)$ is $J\{\underline{z} \to \underline{u}, r\} = r^{m-1}$. Then for $r > 0$, $\underline{u} \in S_m$,

$$f(\underline{u},r) = (2\pi)^{-m/2}\, r^{m-1} \exp(-r^2/2) \qquad (dr\ di_m)$$

and for $r^2 > 0$, $\underline{u} \in S_m$,

$$f(\underline{u},r^2) = \frac{(r^2)^{m/2-1} \exp(-r^2/2)}{2^{m/2}\,\Gamma(m/2)} \cdot \frac{1}{\alpha_m} \qquad (dr^2\ di_m)$$

$$= \frac{(r^2)^{m/2-1} \exp(-r^2/2)}{2^{m/2}\,\Gamma(m/2)} \qquad (dr^2\ dU^{(\nu)}).$$

Since $\underline{u} \in S_m$, we have $\underline{u} \sim U(S_m)$ and $r^2 \sim \chi^2_m$. Further, we see that $\underline{u}$ and r^2 are independent.

1.4 Other Distributions on the Hypersphere

Although the uniform distribution will occupy much of our attention, there are other distributions and results which will prove useful. In this section we will list some of these results and catalog several distributions for later reference.

In the following, a random variable Y will have the beta distribution if the probability density function of Y is of the form,

$$p(y) = \frac{(y-a)^{p-1}\,(b-y)^{q-1}}{B(p,q)\;\;(b-a)^{p+q-1}} \qquad (dy), \qquad a \leq y \leq b,$$

with $p > 0$, $q > 0$ and this will be denoted $Y \sim B_{[a,b]}(p,q)$. Also, unless otherwise indicated, $\nu = m/2 - 1$ for given m.

Theorem 1.4.1: Let $\underline{u} \sim U(S_m)$ and $\underline{\xi} \in S_m$. Then $\underline{u}'\underline{\xi} \sim B_{[-1,1]}(\nu+1/2, \nu+1/2)$.

Proof: By Theorem 1.3.1 we may write $\underline{u}'\underline{\xi} \sim \underline{z}'\underline{\xi}/\|\underline{z}\|$ with $\underline{z} \sim N_m(\underline{0}, I_m)$. Consider a transformation from $\underline{z}$ to $P\underline{z}$ in which P is an orthogonal matrix with first column, $(P)_{\cdot 1}$, equal to $\underline{\xi}$. Then without loss of generality, we may take $\underline{\xi} = (1,0,\ldots,0)'$. Thus $\underline{u}'\underline{\xi} \sim z_1/(z_1^2+\cdots+z_m^2)^{1/2}$. It follows that $(\underline{u}'\underline{\xi})^2 \sim z_1^2/(z_1^2+\cdots+z_m^2) \sim B_{[0,1]}(1/2, \nu+1/2)$. Since $z_1/\|\underline{z}\|$ is distributed symmetrically about zero, we deduce that $\underline{u}'\underline{\xi} \sim B_{[-1,1]}(\nu+1/2, \nu+1/2)$.

Theorem 1.4.1 may be used to demonstrate the pertinence of our choice of measure $U^{(\nu)}$. With $\underline{u}$ and $\underline{\xi}$ as in the preceding theorem, we have

$$P(\underline{u}'\underline{\xi} > t) = \int_t^1 \frac{(x+1)^{\nu-1/2}\ (1-x)^{\nu-1/2}}{B(\nu+1/2,\ \nu+1/2)\ 2^{2\nu}}\,dx$$

$$= \int_t^1 \frac{(1-x^2)^{\nu-1/2}\ \Gamma(2\nu+1)}{\Gamma(\nu+1/2)\ \Gamma(\nu+1/2)\ 2^{2\nu}}\,dx$$

and upon using the Gauss duplication formula for gamma functions,

$$= \int_t^1 \frac{(1-x^2)^{\nu-1/2}\ \Gamma(\nu+1)}{\Gamma(1/2)\ \Gamma(\nu+1/2)}\,dx$$

$$= \int_t^1 \frac{(1-x^2)^{\nu-1/2}}{B(1/2,\ \nu+1/2)}\,dx$$

$$= 1 - W^{(\nu)}(t)$$

where $W^{(\nu)}(t)$ is as defined in section 1.3.

The following remark, due to Saw (1978), will be of frequent use. Let $g(\lambda u)$ satisfy

(1) $g(\lambda u) \geq 0$ a.e. with respect to the measure $W^{(\nu)}(u)$ where $-1 \leq u \leq 1$;

(2) $\int_{-1}^{1} g(\lambda u)\, dW^{(\nu)}(u) = 1.$

Then following Saw (1978), we show that $g(\lambda \underline{u}'\underline{\xi})$ is a density with respect to $U^{(\nu)}(\underline{u})$. Using Theorems 1.3.1 and 1.4.1 we have

$$\int_{S_m} g(\lambda \underline{u}'\underline{\xi})\, dU^{(\nu)}(\underline{u}) = E_{\underline{u}}(g(\lambda \underline{u}'\underline{\xi})), \qquad \underline{u} \sim U(S_m)$$

$$= E_{\underline{z}}(g(\lambda \underline{z}'\underline{\xi}/\|\underline{z}\|)), \qquad \underline{z} \sim N_m(\underline{0}, I_m)$$

$$= E_t(g(\lambda t)), \qquad t \sim B_{[-1,1]}(\nu+1/2, \nu+1/2)$$

$$= \int_{-1}^{1} \frac{g(\lambda t)\ (1-t^2)^{\nu-1/2}}{B(1/2,\ \nu+1/2)}\, dt$$

$$= 1.$$

Hence $g(\lambda \underline{u}'\underline{\xi})$ is also a density on S_m with respect to $U^{(\nu)}$.

Theorem 1.4.2: Let $\underset{\sim}{u} \sim U(S_m)$ and let Q be an m x m symmetric idempotent matrix of rank k. Then $\underset{\sim}{u}'Q\underset{\sim}{u} \sim B_{[0,1]}(k/2,(m-k)/2)$.

Proof: Choose the orthogonal matrix P such that

$$PQP' = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}$$

and by Theorem 1.3.1 write $\underset{\sim}{u} \sim \underset{\sim}{z} / \|\underset{\sim}{z}\|$ where $\underset{\sim}{z} \sim N_m(\underset{\sim}{0},I_m)$. Then $\underset{\sim}{u}'Q\underset{\sim}{u} \sim (P\underset{\sim}{z})'Q(P\underset{\sim}{z}) / \|\underset{\sim}{z}\|^2 = \underset{\sim}{z}'P'QP\underset{\sim}{z} / (\underset{\sim}{z}'\underset{\sim}{z}) = (z_1^2+\cdots+z_k^2) / (z_1^2+\cdots+z_m^2) \sim B_{[0,1]}(k/2,(m-k)/2)$.

Theorem 1.4.3: Let $\underset{\sim}{u} \sim U(S_m)$ and $\underset{\sim}{\xi} \in S_m$. Then $(m-1)^{1/2}\ \underset{\sim}{u}'\underset{\sim}{\xi} / (1-(\underset{\sim}{u}'\underset{\sim}{\xi})^2)^{1/2}$ has a Student's t-distribution with m-1 degrees of freedom (denoted t_{m-1}).

Proof: Again we use Theorem 1.3.1 and as in the proof of Theorem 1.4.1 take $\underset{\sim}{\xi} = (1,0,\ldots,0)'$ without loss of generality. Hence we have, $(m-1)^{1/2}\ \underset{\sim}{u}'\underset{\sim}{\xi}/(1-(\underset{\sim}{u}'\underset{\sim}{\xi})^2)^{1/2} = z_1/[(z_2^2+\cdots+z_m^2)/(m-1)]^{1/2} \sim t_{m-1}$.

In future work we will also have occasion to appeal to a central limit theorem for vector valued random variables which we record here (see Rao (1973)). Let $\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_n$ be independent identically distributed (i.i.d.) vectors in E^m with $E\underset{\sim}{x}_i = \underset{\sim}{\mu}$ and $\mathrm{Cov}(\underset{\sim}{x}_i) = \Sigma$, $1 \leq i \leq n$. If $\underset{\sim}{s} = \underset{\sim}{x}_1+\cdots+\underset{\sim}{x}_n$, then $(\underset{\sim}{s}-n\underset{\sim}{\mu})\ n^{-1/2}$ converges in distribution ($\xrightarrow{d}$) to $N_m(\underset{\sim}{0},\Sigma)$, that is, $(\underset{\sim}{s}-n\underset{\sim}{\mu})\ n^{-1/2} \xrightarrow{d} N_m(\underset{\sim}{0},\Sigma)$.

In subsequent work we shall restrict our attention to those densities on S_m which are of the form $f(\lambda\underset{\sim}{u}'\underset{\sim}{\xi})$ and which satisfy

(1.4.1) (1) $f(\lambda \underline{u}'\underline{\xi})$ is constant on $\underline{u}'\underline{\xi} = t$, that is, $f(\lambda \underline{u}'\underline{\xi})$ is rotationally symmetric about $\underline{\xi}$;

(2) $f(\lambda w)$ is a non-decreasing function of w over $[-1,1]$.

This restriction seems appropriate since many commonly occurring densities arising in practice manifest the conditions (1.4.1). In fact, the conditions (1.4.1) guarantee that $f(\lambda \underline{u}'\underline{\xi})$ is unimodal with mode located at $\underline{\xi}$.

We now introduce several important distributions in directional data.

1.4.1 Langevin Distribution

The Langevin density on S_m is given by

$$r^{(\nu)}(\lambda \underline{u}'\underline{\xi}) = \exp(\lambda \underline{u}'\underline{\xi}) \; / \; \Lambda_\nu(\lambda) \qquad (dU^{(\nu)}(\underline{u})) \tag{1.4.2}$$

where

$$\Lambda_\nu(\lambda) = \int_{-1}^{1} \frac{(1-x^2)^{\nu-1/2}}{B(1/2,\ \nu+1/2)} \exp(\lambda x)\, dx$$

$$= (\lambda/2)^\nu \, \Gamma(\nu+1) \, I_\nu(\lambda)$$

and $I_\nu(\lambda)$ is the modified Bessel function of order ν given in series form as

$$I_\nu(\lambda) = (\lambda/2)^\nu \sum_{k=0}^{\infty} \frac{(\lambda/2)^{2k}}{k! \; \Gamma(\nu+k+1)} .$$

From this we see that $\Lambda_\nu(\lambda)$ may also be expressed in terms of hypergeometric functions as $\Lambda_\nu(\lambda) = {}_0F_1(-;\nu+1;\lambda^2/4)$. The quantities $\underline{\xi}$ and $1/\lambda$ are somewhat analogous to the ordinary mean and variance on the real line.

The cases $m = 2$ and $m = 3$ are of special interest. On the circle, $m = 2$, the Langevin distribution is commonly called the von Mises distribution owing to the fact that von Mises (1918) introduced $r^{(0)}(\lambda\underline{u}'\underline{\xi}) = \exp(\lambda\underline{u}'\underline{\xi}) / I_0(\lambda)$ as a density on the circle in order to study deviations from integral values of measured atomic weights.

Since significant advances for the case $m = 3$ were made by Fisher (1953) while working on a paleomagnetism problem, the Langevin distribution on the sphere is known as the Fisher distribution with density $r^{(1/2)}(\lambda\underline{u}'\underline{\xi}) = (\lambda/\sinh\lambda)\exp(\lambda\underline{u}'\underline{\xi})$. The density $r^{(\nu)}(\lambda\underline{u}'\underline{\xi})$ was generalized to S_m by Watson and Williams (1956). But the first to examine the density $r^{(1/2)}(\lambda\underline{u}'\underline{\xi})$ seems to have been Langevin (1905) in regards to a statistical mechanics study of magnetism. Hence, Watson (1983) has proposed the name "Langevin" distribution to settle any question of authorship priority.

If $\underline{u}$ has density $r^{(\nu)}(\lambda\underline{u}'\underline{\xi})$ we write $\underline{u} \sim R^{(\nu)}(\lambda,\underline{\xi})$. A few notable properties of the Langevin distribution are as follows:

(1) for $\underline{s} = \underline{x}_1+\cdots+\underline{x}_n$ the sum of n i.i.d. unit vectors on S_m, then $\hat{\underline{\xi}} = \underline{s}/\|\underline{s}\|$ is the maximum likelihood estimator of $\underline{\xi}$ if and only if $\underline{x}_i \sim$ i.i.d. $R^{(\nu)}(\lambda,\underline{\xi})$, $1 \leq i \leq n$;

(2) as $\lambda \to 0$, $R^{(\nu)}(\lambda,\underline{\xi})$ tends to the uniform distribution on S_m;

(3) the von Mises distribution on the circle maximizes entropy under the constraint that the mean direction and circular variance are fixed.

It should be noted that the Langevin distribution shares many properties analogous to the normal distribution on the real line. For example, properties (1) and (3) above have analogues in the linear normal case.

1.4.2 Saw's Distribution

A density on S_m introduced by Saw (1978) is defined by

$$(1.4.3) \qquad j^{(\nu)}(\lambda \underset{\sim}{u}'\underset{\sim}{\xi}) = \exp(-\lambda^2) \sum_{k=0}^{\infty} \frac{(2\lambda \underset{\sim}{u}'\underset{\sim}{\xi})^k \, \Gamma(\nu+k/2+1)}{k! \, \Gamma(\nu+1)} \, (dU^{(\nu)}) \, .$$

If $\underset{\sim}{u}$ has density $j^{(\nu)}(\lambda \underset{\sim}{u}'\underset{\sim}{\xi})$ we write $\underset{\sim}{u} \sim J^{(\nu)}(\lambda, \underset{\sim}{\xi})$.

Many interesting properties of the distribution $J^{(\nu)}(\lambda, \underset{\sim}{\xi})$ are given by Saw (1978). Among these properties are the following:

(1) if $s^2 \sim \chi^2_m$ then $j^{(\nu)}(\lambda \underset{\sim}{u}'\underset{\sim}{\xi}) = \exp(-\lambda^2) \, E[\exp(2^{1/2}\lambda s \underset{\sim}{u}'\underset{\sim}{\xi})]$;

(2) as $\lambda \to 0$, $J^{(\nu)}(\lambda, \underset{\sim}{\xi})$ tends to the uniform distribution on S_m;

(3) if $\underset{\sim}{x} \sim N_m(\underset{\sim}{\mu}, I_m)$ and $\underset{\sim}{\mu} \neq \underset{\sim}{0}$, then $\underset{\sim}{u} = \underset{\sim}{x}/\|\underset{\sim}{x}\| \sim J^{(\nu)}(\lambda, \underset{\sim}{\xi})$ where $\underset{\sim}{\xi} = \underset{\sim}{\mu}/\|\underset{\sim}{\mu}\|$ and $\lambda = \|2^{-1/2}\underset{\sim}{\mu}\|$.

1.4.3 Saw's Hyperspherical t-Distribution

Saw (1984) proposed the density on S_m,

$$e_s^{(\nu)}(\lambda \underline{u}'\underline{\xi}) = K_s(\lambda) \; / \; (1-2\lambda \underline{u}'\underline{\xi}+\lambda^2)^{\lambda+s} \qquad (d\underline{U}^{(\nu)}) \tag{1.4.4}$$

for $s = 1,2,\ldots$, where in terms of hypergeometric functions,

$$K_s^{-1}(\lambda) = {}_2F_1(s,\nu+s;\nu+1;\lambda^2)$$

$$= (1-\lambda^2)^{-s} \; {}_2F_1(s,1-s;\nu+1;\lambda^2/(1-\lambda^2)).$$

If $\underline{u}$ has the density $e_s^{(\nu)}(\lambda \underline{u}'\underline{\xi})$ we write $\underline{u} \sim E_s^{(\nu)}(\lambda,\underline{\xi})$.

Some properties discussed by Saw (1984) include

(1) if $\underline{u}|\underline{\xi} \sim E_1^{(\nu)}(\lambda,\underline{\xi})$ and $\underline{\xi} \sim E_1^{(\nu)}(\rho,\underline{\eta})$, then $\underline{u}$ has the Cauchy-like property that $\underline{u} \sim E_1^{(\nu)}(\lambda\rho,\underline{\eta})$;

(2) as $\lambda \to 0$, $E_s^{(\nu)}(\lambda,\underline{\xi})$ tends to the uniform distribution on S_m;

(3) as $s \to \infty$, $E_s^{(\nu)}(\lambda/s,\underline{\xi})$ tends to the Langevin distribution $R^{(\nu)}(\lambda,\underline{\xi})$.

We note that $E_1^{(\nu)}(\lambda,\underline{\xi})$ is the generalization of the Cauchy distribution to S_m; here $e_1^{(\nu)}(\lambda \underline{u}'\underline{\xi}) = (1-\lambda^2)/(1-2\lambda \underline{u}'\underline{\xi}+\lambda^2)^{\nu+1}$. In fact for $s = 1$ and $\nu = 0$, $E_1^{(0)}(\lambda,\underline{\xi})$ is the circular wrapped Cauchy distribution (or essentially the Poisson Kernel from complex analysis).

Now consider the family of densities

$$T = \{ e_s^{(\nu)}(\lambda \underline{u}'\underline{\xi}) : \lambda > 0, \ \underline{\xi} \in S_m, \ s = 0,1,2,\ldots \}$$

which contains the Cauchy distribution on one end of the spectrum for s and the Langevin distribution (analogous to the linear normal) on the other end. Thus we see that the class T shares a property similar to the Student's t-distribution in that T contains distributions analogous to the linear Cauchy and linear normal distributions. It is in this sense that we call a distribution from the class T a "hyperspherical t-distribution."

The following three distributions are given only for the circular case. Due to the need for Gegenbauer polymonials to express these densities on S_m, we postpone the generalizations to S_m until Chapter 2. Rather than describe these circular distributions in Cartesian coordinates, $\underline{x} = (x_1,x_2)' \in S_2$, we use the conventional polar angle notation for circular distributions, $\theta \in (0,2\pi]$, where

$$\theta = \begin{cases} \tan^{-1}(x_2/x_1), & x_1 > 0, \ x_2 \geq 0, \\ \pi/2, & x_1 = 0, \ x_2 = 1, \\ \pi + \tan^{-1}(x_2/x_1), & x_1 < 0, \\ 3\pi/2, & x_1 = 0, \ x_2 = -1, \\ 2\pi + \tan^{-1}(x_2/x_1), & x_1 > 0, \ x_2 < 0, \end{cases}$$

and $-\pi/2 < \tan^{-1}(x_2/x_1) < \pi/2$.

1.4.4 Circular Cardioid Distribution

The circular cardioid distribution, introduced by Jeffreys (1948), has density

$$f_c(\theta) = 1 + 2\lambda\cos(\theta-\theta_0) \qquad (dU^{(0)}) \tag{1.4.6}$$

where $0 < \theta \leq 2\pi$, $0 < \theta_0 \leq 2\pi$ and $|\lambda| < 1/2$. Observe that for $\lambda = 0$ the cardioid distribution reduces to the uniform distribution on S_2.

1.4.5 Circular Point Distribution

The density on S_2 which concentrates all the mass at the point $\theta = \theta_0$ is

$$f_{pt}(\theta) = 1 + 2\sum_{p=1}^{\infty} \cos\, p(\theta-\theta_0) \qquad (dU^{(0)}) \tag{1.4.5}$$

where $0 < \theta \leq \pi$.

1.4.5 Circular Wrapped Normal Distribution

The circular wrapped normal distribution has density

$$f_w(\theta) = (2\pi/\sigma^2)^{1/2} \sum_{k=-\infty}^{\infty} \exp\{\frac{-1}{2\sigma^2}(\theta-\theta_0+2k\pi)^2\} \qquad (dU^{(0)}) \tag{1.4.7}$$

with $\sigma > 0$, $0 < \theta_0 \leq 2\pi$ and $0 < \theta \leq 2\pi$. We may also express

$$f_w(\theta) = 1 + 2 \sum_{k=1}^{\infty} \rho^{k^2} \cos k(\theta-\theta_0) \qquad (dU^{(0)})$$

where $\rho = \exp(-\sigma_2/2)$ and $0 < \theta_0 \leq 2\pi$.

Among properties for the circular wrapped normal distribution are

(1) as $\sigma \to \infty$ ($\rho \to 0$), $f_w(\theta)$ tends to the uniform density on S_2;

(2) as $\sigma \to 0$ ($\rho \to 1$), $f_w(\theta)$ tends to the circular point distribution on $\theta = \theta_0$;

(3) if $\theta_1,\ldots,\theta_n$ are independent wrapped normal random variables, each with mode 0 and respective parameters $\sigma_1^2,\ldots,\sigma_n^2$, then $\theta_1+\cdots+\theta_n \pmod{2\pi}$ has a wrapped normal distribution with parameter $\sigma^2 = \sigma_1^2+\cdots+\sigma_n^2$ and mode 0;

(4) the circular wrapped normal distribution provides the distribution for Brownian motion on the circle;

(5) if $\theta_j \sim$ i.i.d., $j = 1,\ldots,n$, with common distribution function $F(\theta)$, $E(\theta) = 0$ and $-\pi < \theta_j \leq \pi$ for each j, then $s_n = n^{-1/2}(\theta_1+\cdots+\theta_n) \pmod{2\pi}$ tends to the wrapped normal distribution with $\sigma^2 = E(\theta^2)$.

1.5 The Role of the Uniform Distribution

As already remarked in section 1.3, the uniform distribution provides the null hypothesis of no preferred direction for data on S_m. It is in this setting that we are interested in testing for uniformity on S_m. The most prevalent prescriptions for testing uniformity on S_m are discussed in section 1.6.

But beyond being of use in providing the null hypothesis of no preferred direction, the uniform distribution arises in a variety of ways in directional data. Indeed the uniform distribution shares some properties analogous to those of the normal distribution on the real line, albeit most of the important analogous properties are possessed by the Langevin and wrapped normal distributions. In this section we shall examine some characterizations of the uniform distribution.

One property of the normal distribution on the real line due to Geary (1936) is that for $n \geq 2$ i.i.d. observations, the mean and variance are independent if and only if the observations are normally distributed. A similar result holds for observations on S_m.

Let $\underline{x}_1, \ldots, \underline{x}_n$, $(n \geq 2)$, be independent observations on S_m with common density $f(\underline{x})$ which is continuous almost everywhere (a.e.) with respect to Lebesgue measure on S_m. With $\underline{s} = \underline{x}_1 + \cdots + \underline{x}_n$, define the direction, $\bar{\underline{x}}_0$, of the sample mean and the resultant length, r_n, by

$$r_n = \|\underline{s}\| \quad \text{and} \quad \bar{\underline{x}}_0 = \underline{s} / r_n. \tag{1.5.1}$$

The quantities $\bar{\underline{x}}_0$ and $n - r_n$ are somewhat analogous to the sample mean and variance on the real line.

With Lebesgue measure on S_m, B_m as defined in section 1.3 and α_m representing the surface area of S_m, let $b = \text{ess inf}\{\alpha_m f(\underline{x}) : \underline{x} \in S_m\} =$

$\sup\{\inf\{\alpha_m f(\underline{x}) : \underline{x} \in S_m\}\}$ where the supremum is taken over the class of all real valued B_m measurable functions on S_m which are equal to $f(\underline{x})$ a.e. and the infimum is taken over all $\underline{x} \in S_m$. Then Kent, Mardia and Rao (1979) have shown that among all distributions with a density and for $b > 0$, $\bar{\underline{x}}_0$ and r_n are independent if and only if $f(\underline{x})$ is the uniform density on S_m. The rather artificial condition that $b > 0$ is also shown by Kent et al. (1979) to be unnecessary for $m = 2$.

The uniform distribution analogously shares another property with the normal distribution on the real line for it appears as the limiting distribution in the central limit theorem on the circle.

More precisely, let the angles $\theta_1,\cdots,\theta_n$ be i.i.d. random variables with common circular distribution function $F(\theta)$. Then, provided $F(\theta)$ is not the lattice distribution with mass distributed on the vertices of a regular polygon inscribed in the unit circle with no lattice point coinciding to zero, the sum $\theta_1+\cdots+\theta_n \pmod{2\pi}$ converges in distribution to the uniform distribution on the circle. Also under the constraint that the random variables possess axial symmetry, that is, $f(\underline{x}) = f(-\underline{x})$, $\underline{x} \in S_3$, the central limit theorem on the sphere gives the sum of random variables converging to the uniform distribution (see Breitenberger (1963)).

1.6 Testing Uniformity

A fundamental inference problem for data on S_m is testing whether or not the data exhibit a preferred direction, that is, determining whether or not the data originate from a uniform distribution. Besides having practical importance, testing uniformity is of central theoretical interest.

To accommodate applications in which higher dimensional spheres are necessary and more essentially to provide a unified methodology, we examine tests of uniformity which encompass two, three or more dimensions. The bulk of proposed tests of uniformity have been presented in two or three dimensions. Indeed, Mardia (1975) notes, "As in multivariate analysis, it is not surprising that there is a dearth of nonparametric tests of practical value for $p > 3$" (p. 369). In this section we present some conventional tests of uniformity in two or three dimensions which have extensions to higher dimensions, $m > 3$.

Historically, the earliest test of uniformity was based on the vector resultant. The distribution of the resultant was studied by Lord Rayleigh (1880) in regards to a problem in sound concerning the composition of n iso-periodic vibrations of unit amplitude and random phases.

Apparently Rayleigh's solution was not widely known among statisticians when Karl Pearson (1905) posed the problem of the distribution of the resultant length of a man taking n equal length steps from the origin where the direction of each step is chosen at random. Upon learning of Rayleigh's solution for large n, Pearson (1905) colorfully remarked, "The lesson of Lord Rayleigh's solution is that in open country the most probable place to find a drunken man who is at all capable of keeping on his feet is somewhere near his starting point!" (p. 342).

Rayleigh's test of uniformity in m dimensions may be based on an approximation to the distribution of the resultant when the sample size, n, is large (see Watson (1983) for instance).

Suppose $\underline{x}_1,\ldots,\underline{x}_n$ are i.i.d. observations taken from the uniform

distribution on S_m. Let $\underline{\mu} = E\underline{x}$ and $\Sigma = E(\underline{x}-\underline{\mu})(\underline{x}-\underline{\mu})'$. Since $P\underline{x} \sim U(S_m)$ for any orthogonal matrix P, we have $\underline{\mu} = P\underline{\mu}$ so that $\underline{\mu} = \underline{0}$. In addition, since $\Sigma = P\Sigma P'$ and $\text{trace}(\Sigma) = \text{trace}(E\underline{x}\underline{x}') = E\,\text{trace}(\underline{x}'\underline{x}) = 1$ it follows that $\Sigma = I_m/m$. Now with $\underline{s} = \underline{x}_1+\cdots+\underline{x}_n$ and using the central limit theorem for unit vector random variables listed in section 1.4, we find $\underline{s}/n^{1/2} \xrightarrow{d} N_m(\underline{0}, I_m/m)$. Thus $(m/n)^{1/2}\,\underline{s} \xrightarrow{d} N_m(\underline{0}, I_m)$ and putting $r_n = (\underline{s}'\underline{s})^{1/2}$, we may write $(m/n)\,r_n^2$ as the test statistic and reject for large values according to the percentage points of the χ^2_m distribution.

Among tests for uniformity on the circle which have received much attention are those due to B. Ajne (1968). Suppose n points are sampled independently from the circle S_2 with Lebesgue measure. Arbitrarily select an origin on S_2, say Q, and select a positive direction, say counterclockwise. An arc on S_2 of length θ may be expressed as those points on S_2 whose distance from Q belong to the interval $(x, x+\theta]$, $0 \leq x < 2\pi$. Define $N(\theta,x)$ to be the number of observations belonging to the arc corresponding to the interval $(x, x+\theta]$. For example, in Figure 1.1 with n = 7 sample points, we see that $N(\theta,x) = 4$.

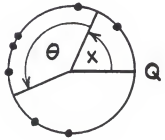

Figure 1.1 Sample Points in an Arc of Length θ on the Circle

Now set

$$(1.6.1) \qquad A(\theta) = \frac{1}{2\pi n} \int_0^{2\pi} \left(N(\theta,x) - \frac{n\theta}{2\pi} \right)^2 dx$$

and

$$(1.6.2) \qquad B(\theta) = \max_{0 \le x \le 2\pi} N(\theta,x).$$

Ajne (1968) studied the statistics $I = 2\pi n\, A(\pi)$ and $N = B(\pi)$ while Rothman (1971) presented slightly more general results for $A(\theta)$ and $B(\theta)$ with $0 < \theta < 2\pi$. Further generalizations have been given by Beran (1968, 1969), Giné (1975), Prentice (1978), Jupp and Spurr (1983) and Saw (1984).

Ajne was concerned with testing the null hypothesis of uniformity

$$H_0: \quad f(\theta) = \frac{1}{2\pi} \qquad 0 < \theta < 2\pi$$

against the alternative

$$H_1: \quad f(\theta) = \begin{cases} q/\pi & 0 < \theta < \pi \\ p/\pi & \pi < \theta < 2\pi \end{cases}$$

where $p + q = 1$ and $0 < p < 1$.

Ajne found that rejecting H_0 for large values of $N = B(\pi)$ provides the most powerful invariant (under rotations) test against "distant" alternatives in H_1 with p near 1. Using a random walk argument, Ajne found the probability distribution of N to be

$$P(N \geq k) = 2^{-(n-1)} (2k - n) \sum_{j=0}^{\infty} \binom{n}{j(2k - n) + k}$$

where $k = [n/2] + 1, \ldots, n$ and $[\cdot]$ is the greatest integer function. These probabilities were tabulated by Ajne (1968) and also by Klotz (1959) as probabilities associated with a bivariate sign test offered by Hodges (1955). The connection between Ajne's test and Hodge's test was first noted by Bhattacharyya and Johnson (1969).

The asymptotic distribution of $N^* = (2N - n) / n^{1/2}$ as the sample size n tends to infinity was given by Ajne (1968) as

$$\lim_{n \to \infty} P(N^* \geq c) = L(\pi/(2c))$$

where L(z) is the limiting distribution of the Kolmogorov–Smirnov statistic D(n), (see Feller (1948)). The power function for N and asymptotic power function for N^* were computed and recorded by Ajne (1968).

Of the two statistics N and I introduced by Ajne, it is the variant $A(\pi) = I/(2\pi n)$ which has been the subject of much generalization. For alternatives in H_1 near uniformity, that is, as $p \to 1/2$ in H_1, Ajne (1968) established that rejecting H_0 for large values of I gives the locally most powerful invariant (under rotations) test.

Using a Fourier expansion technique proposed by Watson (1967), the limiting distribution of $2I/(\pi n)$ as $n \to \infty$ is

$$\frac{4}{\pi^2} \sum_{m=1}^{\infty} \frac{\chi_2^2(\lambda_m)}{(2m - 1)^2}$$

in which $p = 1/2 + xn^{-1/2}$, $x > 0$ and the random variables $\chi_2^2(\lambda_m)$ are

independent chi-square variates with two degrees of freedom and non-centrality parameter $\lambda_m = 16x^2 / [(2m-1)^2\pi^2]$. This limiting distribution has mean $1 + 4x^2$ and variance $2 + 16x^2/3$. Under uniformity; that is, when $x = 0$, the limiting distribution assumes the form

$$P\left(\frac{2I}{n\pi} > c \right) = \frac{4}{\pi} \sum_{j=0}^{\infty} \frac{(-1)^j \exp[-(2j+1)^2 \pi^2 c/8]}{2j+1}, \quad c > 0.$$

Following Prentice (1978), the m-dimensional analogue of Ajne's statistic $I/(2\pi n)$ for a random sample $\underline{x}_1, \ldots, \underline{x}_n$ from S_m (with Lebesgue measure) is

$$A_{m,n} = (n\alpha_m)^{-1} \int_{\underline{\xi}'\underline{\xi}=1} (N(\underline{\xi})-n/2)^2 \, di_m(\underline{\xi}) \tag{1.6.3}$$

where $\alpha_m = 2\pi^{m/2} / \Gamma(m/2)$ is the surface area of the unit m-sphere and $N(\underline{\xi})$ is the number of data points $\underline{x}_i$ such that $\underline{\xi}'\underline{x}_i > 0$. Alternately we may express

$$A_{m,n} = \frac{n}{4} - (\pi n)^{-1} \sum_{i<j} \sum \cos^{-1}(\underline{x}_i'\underline{x}_j) .$$

The asymptotic null distribution of $A_{m,n}$ is

$$\sum_{q=1}^{\infty} a_{2q-1}^2 \, \chi^2_{d(2\nu+1,2q-1)}$$

where

$$a_{2q-1} = \frac{(-1)^{q-1} \, 2^{2\nu} \, \Gamma(\nu+1) \, \Gamma(q+\nu) \, (2q-2)!}{\pi \, (q-1)! \, (2q+2\nu-1)!}$$

and the $\chi^2_{d(2\nu+1,2q-1)}$ random variables are independent with degrees of freedom

$$d(2\nu+1,\ 2q-1) = \binom{2q+2\nu-2}{2\nu} + \binom{2q+2\nu-1}{2\nu}, \quad \nu = \frac{m}{2} - 1 .$$

Beran (1968, 1969) considered generalizations of Ajne's statistic $A(\pi)$ by proposing a class of test statistics on the circle S_2

$$T_n = -\frac{1}{n} \int_0^1 \Big[\sum_{j=1}^{n} f(x+x_j) - n \Big]^2 \, dx \tag{1.6.4}$$

where $f(x)$ is a non-uniform probability density on S_2 (with respect to the measure $U^{(0)}$). Beran showed that in special cases, T_n reduces to Rayleigh's test and Ajne's test among others.

Both Saw (1984) and Prentice (1978) have studied generalizations of the statistics T_n to m-dimensions. Saw (1984) gives this generalization as

$$G_n^{(\nu)} = \int_{S_m} \sum_{j=1}^{n} \Big[\phi(\underline{x}_j'\underline{\xi}) - 1 \Big]^2 \, dU^{(\nu)}(\underline{\xi}) \tag{1.6.5}$$

where $\underline{x}_1, \ldots, \underline{x}_n$ is a random sample with common density $\phi(\underline{x})$ over S_m (with respect to the measure $U^{(\nu)}$). For certain choices of $\phi(\underline{x})$, the statistics $G_n^{(\nu)}$ can be shown to include Rayleigh's test and Ajne's test among others. Using Gegenbauer polynomials (see Chapter 2), it is possible to express $G_n^{(\nu)}$ in summation form. According to Saw (1984),

$$G_n^{(\nu)} = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{p=1}^{\infty} \Big(\bar{\phi}_p^{(\nu)} \Big)^2 A_p^{(\nu)}(\underline{u}_i'\underline{u}_j)$$

in which

$$A_p^{(\nu)}(x) = \frac{\nu + p}{\Gamma(\nu+1)} \sum_{s=0}^{[p/2]} \frac{(-1)^s \, \Gamma(\nu+p-s) \, (2x)^{p-2s}}{s! \, (p-2s)!}, \qquad p \neq 0,$$

$$A_0^{(\nu)}(x) = 1,$$

$$A_p^{(\nu)}(1) \, \bar{\phi}_p^{(\nu)} = \int_{S_m} \phi(\underline{u}'\underline{\xi}) \, A_p{}^{(\nu)}(\underline{u}'\underline{\xi}) \, dU^{(\nu)}(\underline{\xi}).$$

Prentice (1978) also gives the asymptotic distribution of $G_n^{(\nu)}$ under the null hypothesis of uniformity.

Another class of tests which, for the cases $m = 2$ and $m = 3$, includes the tests of Rayleigh, Ajne, Beran and others has been given by Giné (1975). Giné's class of tests is based on Sobolev norms and has since been generalized further by Jupp and Spurr (1983) who considered tests for symmetry also based on Sobolev norms.

CHAPTER 2
GEGENBAUER POLYNOMIALS AND DENSITY REPRESENTATION
ON THE HYPERSPHERE

2.1 Introduction

The representation of densities on S_m will proceed by use of spherical harmonic functions. Thus we present a brief description of some important notions from spherical harmonics.

Let $G_n(\underline{x})$, $\underline{x} \in E^m$ be a homogeneous polynomial of degree n in m dimensions. That is,

$$G_n(\tau\underline{x}) = \tau^n G_n(\underline{x}),$$

$\underline{x} \in E^m$ and τ a non-zero scalar. If $G_n(\underline{x})$ satisfies

$$\nabla_m G_n(\underline{x}) = 0$$

where

$$\nabla_m = \left(\frac{\partial}{\partial \underline{x}}\right)'\left(\frac{\partial}{\partial \underline{x}}\right) = \left(\frac{\partial}{\partial x_1}\right)^2 + \ldots + \left(\frac{\partial}{\partial x_m}\right)^2$$

is the Laplacian operator, then $G_n(\underline{x})$ is called a harmonic function.

With $\underline{u} \in S_m$, define

$$S_n(m,\underline{u}) = G_n(\underline{u}).$$

Müller (1966) calls $S_n(m,\underline{u})$, $\underline{u} \in S_m$, a spherical harmonic function of order n in m dimensions. The spherical harmonic functions are

orthogonal with respect to the measure i_m in that, for $\underline{u} \in S_m$ and $p \neq q$,

$$\int_{S_m} S_p(m,\underline{u})\, S_q(m,\underline{u})\, di_m(\underline{u}) = 0.$$

In addition, there exist $N(m,n) = [(2n+m-2)\ \Gamma(n+m-2)] \ /\ [\Gamma(n+1)\ \Gamma(m-1)]$ linearly independent spherical harmonics $S_{n,j}(m,\underline{u})$, $j = 1,\ldots,N(m,n)$, forming a basis for the spherical harmonic functions of degree n in m dimensions. We may suppose that $S_{n,j}(m,\underline{u})$, $j = 1,\ldots,N(m,n)$ are orthonormal; that is,

$$\int_{S_m} S_{p,j}(m,\underline{u})\, S_{q,j}(m,\underline{u})\, di_m(\underline{u}) = \delta_{pq}$$

where the Kronecker delta, δ_{pq}, is defined to be zero for $p \neq q$ and unity for $p = q$. Thus for any $\underline{u} \in S_m$ we may write

$$S_n(m,\underline{u}) = \sum_{j=1}^{N(m,n)} c_{n,j}\ S_{n,j}(m,\underline{u})$$

for some scalars $\{c_{n,j}\}$ determined by

$$c_{n,j} = \int_{S_m} S_n(m,\underline{u})\, S_{n,j}(m,\underline{u})\, di_m(\underline{u}).$$

However, the basis elements $S_{n,j}(m,\underline{u})$ are complicated to represent for general m-dimensional spheres. To achieve simpler representations we restrict attention to the subset of homogeneous harmonic polynomials of degree n in m dimensions, $P_n(m,\underline{x})$, $\underline{x} \in E^m$, with the property that

$$P_n(m,A\underline{x}) = P_n(m,\underline{x})$$

for all orthogonal transformations A which leave one vector, say $\underline{z}$, fixed. The polynomials $P_n(m,\underline{z})$ are unique up to a multiplicative constant. The constant may be fixed by requiring $P_n(m,\underline{z}) = 1$ where the vector $\underline{z}$ is invariant under the orthogonal transformations described above. Müller (1966) terms $\{P_n(m,\underline{x})\}$ Legendre functions of degree n in m dimensions and shows that every spherical harmonic function $S_n(m,\underline{u})$, $\underline{u} \in S_m$, may be written

$$S_n(m,\underline{u}) = \sum_{j=1}^{N(m,n)} a_j \, P_n(m,\underline{n}_j'\underline{u})$$

where $\underline{n}_1,\ldots,\underline{n}_{N(m,n)}$ are unit vectors with

$$\det\{P_n(m,\underline{n}_i'\underline{n}_j) : i,j = 1,\ldots,N(m,n)\} > 0.$$

We shall be concerned with the expansion of functions in terms of spherical harmonic functions. It is shown by Müller (1966) that every continuous function $F(\underline{u})$, $\underline{u} \in S_m$, can be approximated uniformly by spherical harmonic functions in that,

$$F(\underline{u}) = \lim_{\tau \to 1-0} \sum_{n=0}^{\infty} \tau^n \, S_n(m,\underline{u})$$

where

$$S_n(m,\underline{u}) = \alpha_m^{-1} \, N(m,n) \int_{S_m} P_n(m,\underline{u}'\underline{\xi}) \, F(\underline{\xi}) \, di_m(\underline{\xi})$$

and α_m is the surface area of S_m. Further, Müller (1966) shows that the system of spherical harmonics is complete for continuous functions on S_m in the sense that if

$$\int_{S_m} F(\underline{u})\, S_n(m,\underline{u})\, di_m(\underline{u}) = 0$$

for all spherical harmonics $S_n(m,\underline{u})$, then $F(\underline{u})$ is identically zero.

Fortunately, it suffices for our present purpose, to restrict attention to the subset $\{S_n(m,\underline{\xi})\}$ having the property that $S_n(m,A\underline{\xi}) = S_n(m,\underline{\xi})$ for all orthogonal matrices A such that $A\underline{\xi} = \underline{\xi}$. It will also be important that $P_n(m,\underline{\xi})$, $\underline{\xi} \in S_m$, can be expressed in terms of the Gegenbauer polynomials defined in the next section. Through the use of Gegenbauer polynomials we will achieve representations for densities of the form $f(\lambda \underline{u}'\underline{\xi})$ which are restricted to be rotationally symmetric about $\underline{\xi} \in S_m$; that is, $f(\lambda \underline{u}'\underline{\xi})$ is invariant under orthogonal transformations which leave one point, $\underline{\xi}$, on the m-sphere fixed. In addition we require $f(\lambda \underline{u}'\underline{\xi})$ to be nondecreasing in $\underline{u}'\underline{\xi}$. But before providing such a density decomposition in section 2.3, we examine many of the standard properties of the Gegenbauer polynomials as collected in the next section.

2.2 Gegenbauer Polynomials

We define $\{C_n^{\nu}(x)\}$, $n = 0,1,2,\ldots$ to be a system of polynomials orthogonal on the interval $[-1,1]$ with respect to the weight function $(1-x^2)^{\nu-1/2}$, $\nu > -1/2$; that is, for $m \neq n$,

$$\int_{-1}^{1} C_n^{\nu}(x)\, C_m^{\nu}(x)\, (1-x^2)^{\nu-1/2}\, dx = 0.$$

The weight function $(1-x^2)^{\nu-1/2}$ determines the polynomials $\{C_n^{\nu}(x)\}$ apart from a constant factor. Commonly called Gegenbauer or ultraspherical polynomials, the polynomials $\{C_n^{\nu}(x)\}$ are part of the well known classical orthogonal polynomials along with Hermite, Laguerre, Legendre

and Jacobi polynomials. The notation "$C_n^{\nu}(x)$" is due to Leopold Gegenbauer who studied these polynomials in a series of papers from 1877 to 1893.

Being unique only up to an arbitrary multiplicative constant, orthogonal polynomials are usually standardized in some way. The conventional standardization is to put

$$C_n^{\nu}(1) = \binom{n + 2\nu - 1}{n}, \qquad \nu \neq 0$$

$$C_n^{0}(1) = \frac{2}{n}, \qquad n \neq 0$$

$$C_0^{0}(1) = 1.$$

However, we choose a standardization given by Saw (1984) which lends itself to simplifying certain properties of Gegenbauer polynomials. For example, Saw's choice of standardization and weight function allows the integrated square of a Gegenbauer polynomial to assume the particularly simple form given in (2.2.67).

Saw (1984) defines the polynomials $\{A_p^{(\nu)}(x)\}$ by

$$A_p^{(\nu)}(x) = \frac{\nu + p}{\Gamma(\nu+1)} \sum_{s=0}^{[p/2]} \frac{(-1)^s \, \Gamma(\nu+p-s) \, (2x)^{p-2s}}{s! \, (p-2s)!}, \qquad p \neq 0,$$

$$A_0^{(\nu)}(1) = 1.$$

Equivalently we may define the polynomials $\{A_p^{(\nu)}(x)\}$ by the generating relation given by Saw (1984) as

$$\frac{1 - \lambda^2}{(1 - 2\lambda x + \lambda^2)^{\nu+1}} = \sum_{p=0}^{\infty} A_p^{(\nu)}(x) \lambda^p , \qquad |\lambda| < 1.$$

The polynomials $\{A_p^{(\nu)}(x)\}$ are related to $\{C_p^{\nu}(x)\}$ by

$$A_p^{(\nu)}(x) = \frac{p+\nu}{\nu} C_p^{\nu}(x) , \qquad \nu \neq 0,$$

$$= p\, C_p^0(x) , \qquad p \neq 0,$$

$$= 1 , \qquad \nu = p = 0.$$

Beyond simplifying certain results, the standardization used in the polynomials $\{A_p^{(\nu)}(x)\}$ appears to be quite natural in many ways. For instance, the generating function defining $\{A_p^{(\nu)}(x)\}$ holds for all $\nu > -1/2$, while a separate generating function for $\nu = 0$ is required for the polynomials $\{C_n^{\nu}(x)\}$. Additionally, the factor $1/\nu$ must be included to give the well known result that

$$C_p^0(x) = \lim_{\nu \to 0} (1/\nu)\, C_p^{\nu}(x), \qquad p \neq 0.$$

Yet with Saw's standardization, no additional factors are necessary as is seen from

$$\lim_{\nu \to 0} A_p^{(\nu)}(x) = \lim_{\nu \to 0} \frac{p+\nu}{\nu} C_p^{\nu}(x)$$

$$= \lim_{\nu \to 0} \frac{p}{\nu} C_p^{\nu}(x) + \lim_{\nu \to 0} C_p^{\nu}(x)$$

$$= p\, C_p^0(x)$$

$$= A_p^{(0)}(x), \qquad p \neq 0.$$

For $p = 0$, both $C_0^0(x)$ and $A_0^{(0)}(x)$ are defined to be unity.

The polynomials $\{A_p^{(\nu)}(x)\}$ possess the orthogonality property that for $p \neq q$,

$$\int_{S_m} A_p^{(\nu)}(\underline{u}'\underline{\xi})\, A_q^{(\nu)}(\underline{u}'\underline{\xi})\, dU^{(\nu)}(\underline{u}) = \int_{-1}^{1} A_p^{(\nu)}(x)\, A_q^{(\nu)}(x)\, dW^{(\nu)}(x) = 0.$$

Notice that from the second integral above we see that the polynomials $\{A_p^{(\nu)}(x)\}$ are orthogonal on the interval $[-1,1]$ with respect to the weight function

$$\frac{d}{dx}\, W^{(\nu)}(x) = \frac{(1-x^2)^{\nu-1/2}}{B(1/2,\ \nu+1/2)}.$$

We believe the gain in simplicity and economy using the polynomials $\{A_p^{(\nu)}(x)\}$ is worth the slight modification to the known formulas for Gegenbauer polynomials. Indeed the greatest gain in this redefinition is exhibited by the formula

$$\int_{S_m} A_p^{(\nu)}(\underline{u}'\underline{\xi})\, A_q^{(\nu)}(\underline{u}'\underline{\eta})\, dU^{(\nu)}(\underline{u}) = \begin{cases} 0 & p \neq q, \\ A_p^{(\nu)}(\underline{\xi}'\underline{\eta}) & p = q, \end{cases}$$

given by Saw (1984). Henceforth we will use the polynomials $\{A_p^{(\nu)}(x)\}$ exclusively, rather than the traditional polynomials $\{C_n^{\nu}(x)\}$. The only differences between the polynomials $\{C_n^{\nu}(x)\}$ and $\{A_p^{(\nu)}(x)\}$ occur in the respective standardizations and the fact that the weight functions

differ by a constant factor. Still, the two sets of polynomials share the same basic orthogonality relationship, so that we shall also refer to $\{A_p^{(\nu)}(x)\}$ as Gegenbauer polynomials.

Since we will require extensive use of the properties of Gegenbauer polynomials in the sequel and since it is cumbersome to collect and continually modify certain formulas to our notation, we record here a list of many standard properties in terms of the notation "$A_p^{(\nu)}(x)$." For definitions of other notations used in this section, see the Appendix listing the index of notations. Also for a further account of the results given in this section and for other results, see Abramowitz and Stegun (1972), Carlson (1977), Durand, Fishbane and Simmons (1976), Erdélyi (1953), Gegenbauer (1877, 1884, 1890, 1893), Hsü (1938), Mehdi and Borah (1984), Rainville (1960), Robin (1957), Saw (1984) and Szego (1975).

2.2.1 Asymptotic Relations

$$(2.2.1)\quad A_p^{(\nu)}(\cos\theta) = \frac{2(p,\nu+1)}{p!} \sum_{m=0}^{M-1} \frac{(\nu,m)(1-\nu,m)}{m!\,(p+\nu-m,m)} \times \frac{\cos[(p+\nu-m)\theta - (\nu+m)\pi/2]}{(2\sin\theta)^{\nu+m}} + O(p^{-M-1/2})$$

$$0 < \theta < \pi,\ \nu \neq 0,-1,-2,\ldots$$

$$(2.2.2)\quad A_p^{(\nu)}(\cos\theta) = \frac{p!\,A_p^{(\nu)}(1)}{B(\nu,\nu+1)} \sum_{m=0}^{M-1} \frac{(1-\nu,m)}{m!\,(\nu+m,p+1)} \times \frac{\cos[(p+\nu+m)\theta - (\nu+m)\pi/2]}{(2\sin\theta)^{\nu+m}} + R_M(\theta)$$

$$0 < \nu < 1,\ 0 < \theta < \pi,$$

$$|R_M(\theta)| \le \frac{p!\,A_p^{(\nu)}(1)}{B(\nu,\nu+1)} \frac{(1-\nu,M)}{M!\,(M+\nu,p+1)} \frac{A}{(2\sin\theta)^{\nu+M}}$$

$$A = \begin{cases} 2\sin\theta & \text{if } \sin^2\theta \ge 1/2 \\ |\cos\theta|^{-1} & \text{if } \sin^2\theta < 1/2 \end{cases}$$

$$(2.2.3)\quad A_p^{(\nu)}(\cos\theta) \sim \frac{(p/2)^{\nu-1}\,(p+\nu)}{\Gamma(\nu+1)\,\sin^{\nu}\theta} \cos[(p+\nu)\theta - \nu\pi/2]$$

as $p \to \infty$, $\sin\theta \neq 0$

2.2.2 Christoffel-Darboux Formula

$$(2.2.4)\quad 2(x-y)\sum_{p=0}^{n} \frac{A_p^{(\nu)}(x)\,A_p^{(\nu)}(y)}{A_p^{(\nu)}(1)} = \frac{(n+1)}{(\nu+n+1)\,A_n^{(\nu)}(1)} [A_{n+1}^{(\nu)}(x)\,A_n^{(\nu)}(y) - A_n^{(\nu)}(x)\,A_{n+1}^{(\nu)}(y)],\qquad n \neq -2\nu$$

2.2.3 Differential Equations

$$a(x)y'' + b(x)y' + c(x) = 0$$

	y	a(x)	b(x)	c(x)
(2.2.5)	$A_p^{(\nu)}(x)$	$1-x^2$	$-(2\nu+1)x$	$p(p+2\nu)$
(2.2.6)	$D^n A_p^{(\nu)}(x)$	$1-x^2$	$-(2\nu+2n+1)x$	$(p-n)(p+2\nu+n)$
(2.2.7)	$(1-x^2)^{\nu-1/2} \times A_p^{(\nu)}(x)$	$1-x^2$	$(2\nu-3)x$	$(p+1)(p+2\nu-1)$
(2.2.8)	$\sin^\nu x\, A_p^{(\nu)}(x)$	1	0	$(p+\nu)^2 + \frac{\nu(1-\nu)}{\sin^2 x}$
(2.2.9)	$(1-x^2)^{\nu/2-1/4} \times A_p^{(\nu)}(x)$	1	0	$\frac{(p+\nu)^2}{1-x^2} + \frac{2+4\nu-4\nu^2+x^2}{4(1-x^2)^2}$

2.2.4 Differential Relations

$$(2.2.10)\quad D^n A_p^{(\nu)}(x) = 2^n\,(\nu+1,n)\,A_{p-n}^{(\nu+n)}(x)$$

$$(2.2.11)\quad DA_p^{(\nu)}(0) = \begin{cases} 2\,(p+\nu)\,(\nu+1,m)\,(-1)^m/m! & p = 2m+1 \\ 0 & p = 2m \end{cases}$$

$$(2.2.12)\quad \nu p!(-1)^p\, A_p^{(\nu)}[x(x^2-1)^{-1/2}] = (p+\nu)(x^2-1)^{\nu+p/2}\, D^p[(x^2-1)^{-\nu}]$$

$$(2.2.13)\quad x(p+\nu-1)\,DA_p^{(\nu)}(x) - (p+\nu)\,DA_{p-1}^{(\nu)}(x) = p(p+\nu-1)\,A_p^{(\nu)}(x)$$

$$(2.2.14)\quad (p+\nu-1)(x^2-1)\,DA_p^{(\nu)}(x) = p(p+\nu-1)xA_p^{(\nu)}(x) - (p+\nu)(p+2\nu-1)A_{p-1}^{(\nu)}(x)$$

(2.2.15) $(p+\nu-1)DA_{p+1}^{(\nu)}(x) - (p+\nu+1)DA_{p-1}^{(\nu)}(x) = 2[(p+\nu)^2-1]A_p^{(\nu)}(x)$

(2.2.16) $(p+\nu)DA_{p+1}^{(\nu)}(x) - (p+\nu+1)xDA_p^{(\nu)}(x) = (p+2\nu)(p+\nu+1)A_p^{(\nu)}(x)$

(2.2.17) $(p+\nu+1)(1-x^2)\,DA_p^{(\nu)}(x) = (p+2\nu)(p+\nu+1)xA_p^{(\nu)}(x) - (p+1)(p+\nu)A_{p+1}^{(\nu)}(x)$

(2.2.18) $(x-1)[(p+\nu-1)DA_p^{(\nu)}(x) + (p+\nu)A_{p-1}^{(\nu)}(x)] = p(p+\nu-1)A_p^{(\nu)}(x) - (p+\nu)(p+2\nu-1)A_{p-1}^{(\nu)}(x)$

(2.2.19) $(x-1)\,DA_p^{(\nu)}(x) = pA_p^{(\nu)}(x) + 2(p+\nu)\sum_{k=0}^{p-1}(-1)^{p-k}A_k^{(\nu)}(x)$

(2.2.20) $pA_p^{(\nu)}(x) + (1-x)\,DA_p^{(\nu)}(x) = 2(p+\nu)\sum_{k=0}^{p-1}\frac{1}{p+k}\left[\nu A_k^{(\nu)}(x) + (x-1)\,DA_{p+1}^{(\nu)}(x)\right]$

2.2.5 Explicit Representations

(2.2.21) $A_p^{(\nu)}(x) = \dfrac{(p+\nu)}{\Gamma(\nu+1)}\sum_{k=0}^{[p/2]}\dfrac{(-1)^k\,\Gamma(p+\nu-k)}{k!\,(p-2k)!}(2x)^{p-2k}, \quad p \neq 0$

(2.2.22) $A_p^{(\nu)}(x) = p!\,A_p^{(\nu)}(1)\sum_{k=0}^{[p/2]}\dfrac{(1/2,k)\,x^{p-2k}\,(x^2-1)^k}{(2k)!\,(p-2k)!\,(\nu+1/2,k)}, \quad p \neq 0$

(2.2.23) $A_p^{(\nu)}(x) = p!\,A_p^{(\nu)}(1)\sum_{k=0}^{p}\dfrac{(p+2\nu,k)\,[(x-1)/2]^k}{k!\,(p-k)!\,(\nu+1/2,k)}, \quad p \neq 0$

(2.2.24) $A_p^{(\nu)}(x) = p!\,2^{-p}\,A_p^{(\nu)}(1)\sum_{k=0}^{p}\dfrac{(x-1)^k\,(x+1)^{p-k}}{k!\,(p-k)!\,(\nu+1/2,k)\,(p+\nu+1/2,-k)}, \quad p \neq 0$

$$(2.2.25)\quad A_p^{(\nu)}(x) = \frac{(p+\nu)}{\Gamma(\nu+1)} \sum_{k=0}^{p} \frac{\Gamma(\nu+k)\ (\nu,p-k)}{k!\ (p-k)!} \left[\ x-(x^2-1)^{1/2}\ \right]^k \times\left[x+(x^2-1)^{1/2}\right]^{p-k}, \qquad p \neq 0$$

$$(2.2.26)\quad A_0^{(\nu)}(x) = 1$$

$$(2.2.27)\quad A_p^{(\nu)}(\cos\theta) = \frac{p+\nu}{\nu} \sum_{k=0}^{p} \frac{(\nu,k)\ (\nu,p-k)}{k!\ (p-k)!} \cos(p-2k)\theta\ , \qquad \nu \neq 0$$

$$(2.2.28)\quad A_p^{(\nu)}(\cos\theta) = \frac{A_p^{(\nu)}(1)}{B(\nu,\nu+1)} \sum_{k=0}^{\infty} \frac{(p+2\nu,k)\ \cos\left[(p+2\nu+2k)\theta-\nu\pi\right]}{k!\ (\nu+k,p+1)}\ ,$$

$$0 < \nu < 1,\ 0 < \theta < \pi$$

$$(2.2.29)\quad A_p^{(0)}(\cos\theta) = 2\cos p\theta, \qquad p = 1,2,\ldots$$

$$(2.2.30)\quad A_p^{(1)}(\cos\theta) = \frac{(p+1)\sin(p+1)\theta}{\sin\theta}\ , \qquad p = 0,1,\ldots$$

$$(2.2.31)\quad A_p^{(\nu)}(0) = \begin{cases} (-1)^{p/2}\ A_{p/2}^{(\nu/2)}(1) & p = 0,2,4,\ldots \\ & (p,\nu) \neq (0,0) \\ 0 & p = 1,3,5,\ldots \end{cases}$$

$$(2.2.32)\quad A_p^{(\nu)}(1) = \begin{cases} \dfrac{2(p+\nu)\ \Gamma(p+2\nu)}{p!\ \Gamma(2\nu+1)} & (p,\nu) \neq (0,0) \\ 1 & p = 0 \\ 2 & \nu = 0,\ p = 1,2,\ldots \end{cases}$$

Table 2.1 Gegenbauer Polynomials $A_p^{(0)}(x)$ on S_2 for $1 \leq p \leq 10$

$$A_1^{(0)}(x) = 2x$$

$$A_2^{(0)}(x) = 4x^2 - 2$$

$$A_3^{(0)}(x) = 8x^3 - 6x$$

$$A_4^{(0)}(x) = 16x^4 - 16x^2 + 2$$

$$A_5^{(0)}(x) = 32x^5 - 40x^3 + 10x$$

$$A_6^{(0)}(x) = 64x^6 - 96x^4 + 36x^2 - 2$$

$$A_7^{(0)}(x) = 128x^7 - 224x^5 + 112x^3 - 14x$$

$$A_8^{(0)}(x) = 256x^8 - 512x^6 + 320x^4 - 64x^2 + 2$$

$$A_9^{(0)}(x) = 512x^9 - 1152x^7 + 864x^5 - 240x^3 + 18x$$

$$A_{10}^{(0)}(x) = 1024x^{10} - 2560x^8 + 2240x^6 - 800x^4 + 100x^2 - 2$$

Table 2.2 Gegenbauer Polynomials $A_p^{(1/2)}(x)$ on S_3 for $1 \leq p \leq 10$

$$A_1^{(1/2)}(x) = 3x$$

$$A_2^{(1/2)}(x) = \frac{5}{2}\left(3x^2 - 1\right)$$

$$A_3^{(1/2)}(x) = \frac{7}{2}\left(5x^3 - 3x\right)$$

$$A_4^{(1/2)}(x) = \frac{9}{8}\left(35x^4 - 30x^2 + 3\right)$$

$$A_5^{(1/2)}(x) = \frac{11}{8}\left(63x^5 - 70x^3 + 15x\right)$$

$$A_6^{(1/2)}(x) = \frac{13}{16}\left(231x^6 - 315x^4 + 105x^2 - 5\right)$$

$$A_7^{(1/2)}(x) = \frac{15}{16}\left(429x^7 - 693x^5 + 315x^3 - 35x\right)$$

$$A_8^{(1/2)}(x) = \frac{17}{128}\left(6435x^8 - 12012x^6 + 6930x^4 - 1260x^2 + 35\right)$$

$$A_9^{(1/2)}(x) = \frac{19}{128}\left(12155x^9 - 25740x^7 + 18018x^5 - 4620x^3 + 315x\right)$$

$$A_{10}^{(1/2)}(x) = \frac{21}{256}\left(46189x^{10} - 109395x^8 + 90090x^6 - 30030x^4 + 3465x^2 - 63\right)$$

Table 2.3 Coefficients a_s for Gegenbauer Polynomials $A_p^{(\nu)}(x)$ on S_m, $1 \leq p \leq 10$

$$A_p^{(\nu)}(x) = (\nu + p) \sum_{s=0}^{p} a_s x^s \qquad (\nu + 1)_n = (\nu + 1)(\nu + 2)\cdots(\nu + n)$$

p	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}
1		2									
2	-1		$2(\nu+1)$								
3		$-2(\nu+1)$		$\frac{4}{3}(\nu+1)_2$							
4	$\frac{1}{2}(\nu+1)$		$-2(\nu+1)_2$		$\frac{2}{3}(\nu+1)_3$						
5		$(\nu+1)_2$		$\frac{-4}{3}(\nu+1)_3$		$\frac{4}{15}(\nu+1)_4$					
6	$\frac{-1}{6}(\nu+1)_2$		$(\nu+1)_3$		$\frac{-2}{3}(\nu+1)_4$		$\frac{4}{45}(\nu+1)_5$				
7		$\frac{-1}{3}(\nu+1)_3$		$\frac{2}{3}(\nu+1)_4$		$\frac{-4}{15}(\nu+1)_5$		$\frac{8}{315}(\nu+1)_6$			
8	$\frac{1}{24}(\nu+1)_3$		$\frac{-1}{3}(\nu+1)_4$		$\frac{1}{3}(\nu+1)_5$		$\frac{-4}{45}(\nu+1)_6$		$\frac{2}{315}(\nu+1)_7$		
9		$\frac{1}{12}(\nu+1)_4$		$\frac{-2}{9}(\nu+1)_5$		$\frac{2}{15}(\nu+1)_6$		$\frac{-8}{315}(\nu+1)_7$		$\frac{4}{2835}(\nu+1)_8$	
10	$\frac{-1}{120}(\nu+1)_4$		$\frac{1}{12}(\nu+1)_5$		$\frac{-1}{9}(\nu+1)_6$		$\frac{2}{45}(\nu+1)_7$		$\frac{-2}{315}(\nu+1)_8$		$\frac{4}{14175}(\nu+1)_9$

2.2.6 Generating Functions

$$(2.2.33)\quad \sum_{p=0}^{\infty} A_p^{(\nu)}(x)\,\lambda^p = \frac{1-\lambda^2}{(1-2\lambda x+\lambda^2)^{\nu+1}}\,, \qquad |\lambda| < 1$$

$$(2.2.34)\quad \sum_{p=0}^{\infty} \frac{\nu}{p+\nu} A_p^{(\nu)}(x)\,\lambda^p = (1-2\lambda x+\lambda^2)^{-\nu}\,, \qquad \nu \neq 0,\ |\lambda| < 1$$

$$(2.2.35)\quad \sum_{p=0}^{\infty} \frac{\nu}{p+\nu} A_p^{(\nu)}(x)\,\lambda^p = (1-\lambda)^{-2\nu}\ {}_1F_0\left[\nu;-;\frac{-2\lambda(1-x)}{(1-\lambda)^2}\right],$$

$$\nu \neq 0,\ |\lambda| < 1$$

$$(2.2.36)\quad 1 + \sum_{p=1}^{\infty} \frac{1}{p} A_p^{(0)}(x)\,\lambda^p = -\ln(1-2\lambda x+\lambda^2)\,, \qquad |\lambda| < 1$$

$$(2.2.37)\quad \sum_{p=0}^{\infty} \frac{(\nu+1/2,p)}{p!} \frac{A_p^{(\nu)}(x)}{A_p^{(\nu)}(1)}\,\lambda^p = r^{-1}\left[(1-\lambda x+r)/2\right]^{1/2-\nu},$$

$$\nu \neq 0,\ |\lambda| < 1,\ r = (1-2\lambda x+\lambda^2)^{1/2}$$

$$(2.2.38)\quad \sum_{p=0}^{\infty} \frac{A_p^{(\nu)}(\cos\theta)}{A_p^{(\nu)}(1)} \frac{\lambda^p}{p!} = \frac{\Gamma(\nu+1/2)\,\exp(\lambda\cos\theta)}{\left(\frac{\lambda}{2}\sin\theta\right)^{\nu-1/2}}\,J_{\nu-1/2}(\lambda\sin\theta)\,,$$

$$\nu \neq 0$$

$$(2.2.39)\quad \sum_{p=0}^{\infty} \frac{A_p^{(\nu)}(\cos\theta)}{A_p^{(\nu)}(1)} \frac{\lambda^p}{p!} = \exp(\lambda\cos\theta)\ {}_0F_1\left[-;\nu+1/2;-\frac{\lambda^2}{4}\sin^2\theta\right],$$

$$\nu \neq 0$$

$$(2.2.40)\quad \sum_{p=0}^{\infty} \frac{(\alpha,p)}{p!}\,\frac{A_p^{(\nu)}(x)}{A_p^{(\nu)}(1)}\,\lambda^p = (1-\lambda x)^{-\alpha}\,{}_2F_1\Big[\begin{matrix}\alpha & \alpha+1\\ 2 & 2\end{matrix};\nu+1/2;\frac{\lambda^2(x^2-1)}{(1-\lambda x)^2}\Big]$$

$$(2.2.41)\quad \sum_{p=0}^{\infty} \frac{A_p^{(\nu)}(x)}{(\nu+1/2,p)\,A_p^{(\nu)}(1)}\,\frac{\lambda^p}{p!} = {}_0F_1\Big[-;\nu+1/2;\frac{\lambda(x-1)}{2}\Big] \times {}_0F_1\Big[-;\nu+1/2;\frac{\lambda(x+1)}{2}\Big]$$

$$(2.2.42)\quad \sum_{p=0}^{\infty} \frac{(\alpha,p)\,(2\nu-\alpha,p)}{p!\,(\nu+1/2,p)}\,\frac{A_p^{(\nu)}(x)}{A_p^{(\nu)}(1)}\,\lambda^p = {}_2F_1\Big[\alpha,2\nu-\alpha;\nu+1/2;\frac{1-\lambda-r}{2}\Big] \times {}_2F_1\Big[\alpha,2\nu-\alpha;\nu+1/2;\frac{1+\lambda-r}{2}\Big],\qquad r=(1-2\lambda x+\lambda^2)^{1/2}$$

2.2.7 Hypergeometric Representations

$$(2.2.43)\quad A_p^{(\nu)}(x) = A_p^{(\nu)}(1)\,{}_2F_1\Big[-p,p+2\nu;\nu+1/2;\frac{1-x}{2}\Big]$$

$$(2.2.44)\quad A_p^{(\nu)}(x) = (-1)^p\,A_p^{(\nu)}(1)\,{}_2F_1\Big[-p,p+2\nu;\nu+1/2;\frac{1+x}{2}\Big]$$

$$(2.2.45)\quad A_p^{(\nu)}(x) = \frac{2^p\,(\nu+1,p)}{p!}\,(x-1)^p\,{}_2F_1\Big[-p,1/2-p-\nu;1-2p-2\nu;\frac{2}{1-x}\Big]$$

$$(2.2.46)\quad A_p^{(\nu)}(x) = A_p^{(\nu)}(1)\,\Big(\frac{x+1}{2}\Big)^p\,{}_2F_1\Big[-p,1/2-p-\nu;\nu+1/2;\frac{x-1}{x+1}\Big]$$

$$(2.2.47)\quad A_p^{(\nu)}(x) = \frac{(\nu+1,p)\ (2x)^p}{p!}\ {}_2F_1\left[\ \frac{-p}{2}\ ,\ \frac{1-p}{2}\ ;1-p-\nu;\frac{1}{x^2}\ \right]$$

$$(2.2.48)\quad A_p^{(\nu)}(x) = A_p^{(\nu)}(1)\ x^p\ {}_2F_1\left[\ \frac{-p}{2},\frac{1-p}{2};\nu+1/2;\frac{x^2-1}{x^2}\ \right]$$

$$(2.2.49)\quad A_p^{(\nu)}(x) = \frac{(\nu+1,p)\left\{x+(x^2-1)^{1/2}\right\}^p}{p!}\ {}_2F_1\left[-p,\nu;1-p-\nu;\left\{x-(x^2-1)^{1/2}\right\}^2\right]$$

$$(2.2.50)\quad A_{2p}^{(\nu)}(x) = (-1)^p\ A_p^{(\nu/2)}(1)\ {}_2F_1\left[\ -p,p+\nu;1/2;x^2\ \right]$$

$$(2.2.51)\quad A_{2p}^{(\nu)}(x) = A_{2p}^{(\nu)}(1)\ {}_2F_1\left[\ -p,p+\nu;\nu+1/2;1-x^2\ \right]$$

$$(2.2.52)\quad A_{2p+1}^{(\nu)}(x) = \frac{(-1)^p}{p!}\ (2p+\nu+1)(\nu+1,p)(2x)\,{}_2F_1\left[-p,p+\nu+1;3/2;x^2\right]$$

$$(2.2.53)\quad A_{2p+1}^{(\nu)}(x) = xA_{2p+1}^{(\nu)}(1)\ {}_2F_1\left[-p,p+\nu+1;\nu+1/2;1-x^2\ \right]$$

2.2.8 Inequalities

$$(2.2.54)\quad \left|A_p^{(\nu)}(x)\right| \leq \max_{-1\leq x\leq 1}\left|A_p^{(\nu)}(x)\right| = A_p^{(\nu)}(1), \qquad \nu > 0,\ -1 \leq x \leq 1$$

$$(2.2.55)\quad \left|A_{2p}^{(\nu)}(x)\right| \leq \max_{-1\leq x\leq 1}\left|A_{2p}^{(\nu)}(x)\right| = \left|A_{2p}^{(\nu)}(0)\right|$$

$$-1 \leq x \leq 1,\ -p < \nu < 0,\ \nu \text{ not an integer}$$

$$(2.2.56)\quad \left|A_p^{(\nu)}(\cos\theta)\right| < (p/2)^{\nu-1}\ \frac{(p+\nu)}{\Gamma(\nu+1)\ \sin^{\nu}\theta}\ ,\ 0 < \nu < 1,\ 0 \leq \theta \leq \pi$$

$$(2.2.57)\quad \max_{-1\le x\le 1} \left|A^{(\nu)}_{2p+1}(x)\right| < 2/p!\ \left[(2p+1)(2p+2\nu+1)\right]^{-1/2} \times \left|(2p+\nu+1)(\nu+1,p)\right|,\quad -p-1/2 < \nu < 0,\ \nu \text{ not an integer}$$

2.2.9 Integral Representations

$$(2.2.58)\quad A^{(\nu)}_p(x) = \frac{A^{(\nu)}_p(1)}{B(1/2,\nu)} \int_0^{\pi} \left[\, x+(x^2-1)^{1/2}\cos\phi \,\right]^p (\sin\phi)^{2\nu-1} d\phi \qquad \nu > 0$$

$$(2.2.59)\quad A^{(\nu)}_p(\cos\theta) = \frac{2^{\nu} A^{(\nu)}_p(1)\ (\sin\theta)^{1-2\nu}}{B(1/2,\nu)} \int_0^{\theta} \frac{\cos(p+\nu)\phi}{(\cos\phi - \cos\theta)^{1-\nu}}\, d\phi \qquad \nu > 0,\ 0 < \theta < \pi$$

2.2.10 Integrals of Gegenbauer Polynomials

$$(2.2.60)\quad \int_a^b A^{(\nu)}_p(x)\, dx = \frac{\left[(p+\nu-1)A^{(\nu)}_{p+1}(x) - (p+\nu+1)A^{(\nu)}_{p-1}(x)\right]_a^b}{2\left[(p+\nu)^2 - 1\right]} \qquad (p+\nu)^2 \neq 1$$

$$(2.2.61)\quad \int_{-1}^{1} A^{(\nu)}_p(x)\, dW^{(\nu)}(x) = \begin{cases} 1 & p = 0 \\ 0 & p = 1,2,\ldots \end{cases}$$

$$(2.2.62)\quad \int_{-1}^{x} A^{(\nu)}_p(y)\, dW^{(\nu)}(y) = \frac{-(2\nu+1)}{p(p+2\nu)} A^{(\nu+1)}_{p-1}(x)\ w^{(\nu+1)}(x)$$

$$p = 1,2,\ldots,\quad w^{(\nu)}(x) = \frac{d}{dx} W^{(\nu)}(x) = (1-x^2)^{\nu-1/2} / B(1/2,\nu+1/2)$$

$$(2.2.63)\quad \int_0^x A_p^{(\nu)}(y)\, dW^{(\nu)}(y) = \frac{1}{p(p+2\nu)} \left[2(\nu+1)w^{(\nu)}(0)A_{p-1}^{(\nu+1)}(0) - (2\nu+1)w^{(\nu+1)}(x)A_{p-1}^{(\nu+1)}(x)\right],\quad p = 1,2,\ldots$$

$$(2.2.64)\quad \int_{-1}^{1} A_p^{(\nu)} \left[\left(\frac{1-x}{2}\right)\cos(\theta+\phi) + \left(\frac{1+x}{2}\right)\cos(\theta-\phi)\right] dW^{(\nu-1/2)}(x) = A_p^{(\nu)}(\cos\theta)\, A_p^{(\nu)}(\cos\phi) \,/\, A_p^{(\nu)}(1)$$

$$(2.2.65)\quad \int_0^{\pi} \exp(\imath z\cos\theta)\, A_p^{(\nu)}(\cos\theta)\, (\sin\theta)^{2\nu}\, d\theta = \pi 2^{1-\nu}\, (\nu,\nu)\, A_p^{(\nu)}(1)\, \imath^p\, z^{-\nu}\, J_{p+\nu}(z)$$

$$(2.2.66)\quad \int_{-1}^{1} A_p^{(\nu)}(x)\, A_q^{(\nu)}(x)\, dW^{(\nu)}(x) = \begin{cases} A_p^{(\nu)}(1) & p = q \\ 0 & p \neq q \end{cases}$$

$$(2.2.67)\quad \int_{S_m} A_p^{(\nu)}(\underline{\alpha}'\underline{u})A_q^{(\nu)}(\underline{\beta}'\underline{u})dW^{(\nu)}(\underline{u}) = \begin{cases} A_p^{(\nu)}(\underline{\alpha}'\underline{\beta}) & p = q \\ 0 & p \neq q \end{cases}$$

$$(2.2.68)\quad (p-q)(p+q+2\nu) \int_a^b A_p^{(\nu)}(x)\, A_q^{(\nu)}(x)\, dW^{(\nu)}(x) = \frac{2\nu+1}{2(\nu+1)} \left[w^{(\nu+1)}(x) \left\{A_p^{(\nu)}(x)DA_q^{(\nu)}(x) - A_q^{(\nu)}(x)DA_p^{(\nu)}(x)\right\}\right]_a^b,\quad p \neq q$$

$$(2.2.69)\quad I^{(\nu)}(p,q,r) = \int_{-1}^{1} A_p^{(\nu)}(x)A_q^{(\nu)}(x)A_r^{(\nu)}(x)\, dW^{(\nu)}(x) =$$

$$\frac{(p+\nu)(q+\nu)(r+\nu)B(\nu,s)}{\nu^2\,(s+\nu)\,B(2\nu,s)} \binom{s-p+\nu-1}{s-p} \binom{s-q+\nu-1}{s-q} \binom{s-r+\nu-1}{s-r}$$

$$\nu > 0,\ p \leq q \leq r,\ p+q \geq r,\ p+q+r = 2s \text{ is even}$$

$$(2.2.70)\quad I^{(0)}(p,q,r) = \begin{cases} 0 & p \leq q \leq r,\ p+q \neq r \\ 2 & p \leq q \leq r,\ p+q = r \end{cases}$$

$$(2.2.71)\quad I^{(1)}(p,q,r) = \begin{cases} (p+1)(q+1)(r+1) & p \leq q \leq r,\ p+q \geq r \\ & p+q+r \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

$$(2.2.72)\quad I^{(\nu)}(1,q,r) = \begin{cases} \dfrac{r(\nu+1)}{\nu+r} A_r^{(\nu)}(1) & q = r-1 \\ \dfrac{(\nu+1)(r+2\nu)}{\nu+r} A_r^{(\nu)}(1) & q = r+1 \\ 0 & |q-r| > 1 \end{cases}$$

$$(2.2.73)\quad I^{(\nu)}(p,q,r) = \frac{p+\nu}{p}\Big[\frac{q+1}{q+\nu+1} I^{(\nu)}(p-1,q+1,r)$$

$$+ \frac{q+2\nu-1}{q+\nu-1} I^{(\nu)}(p-1,q-1,r) - \frac{p+2\nu-2}{p+\nu-2} I^{(\nu)}(p-2,q,r)\Big],$$

$$p \geq 2,\ q \geq 1,\ r \geq 0$$

2.2.10 Miscellaneous Relations

$$(2.2.74)\quad A_p^{(\nu)}(-x) = (-1)^p A_p^{(\nu)}(x)$$

$$(2.2.75)\quad A_p^{(\nu)}(x) = \frac{(-1/2)^p p!\,(2p+2\nu-1)(x^2-1)^{p/2}}{(2\nu-1)\ (\nu+1/2,p)}\ A_p^{(\nu)}(1) \times A_p^{(1/2-p-\nu)}(x(x^2-1)^{-1/2})$$

$$(2.2.76)\quad A_p^{(\nu)}(\cos\theta\cos\phi + x\sin\theta\sin\phi) = \sum_{k=0}^{p} \frac{(\nu+1,k)(\sin\theta\sin\phi)^k}{(\nu+1/2,k)\ A_{p-k}^{(\nu+k)}(1)}\ A_{p-k}^{(\nu+k)}(\cos\theta)\ A_{p-k}^{(\nu+k)}(\cos\phi)\ A_k^{(\nu-1/2)}(x)$$

$$(2.2.77)\quad \sum_{p=1}^{n} A_p^{(0)}(x) = \frac{A_{2n}^{(1)}\left(\left[\ \frac{x+1}{2}\ \right]^{1/2}\right)}{2n+1} - 1$$

$$(2.2.78)\quad \sum_{p=0}^{n} \frac{A_{2p}^{(1)}(x)}{2p+1} = \left[\ A_n^{(1)}(x)\ /\ (n+1)\ \right]^2$$

2.2.11 Recurrence Relations

$$(2.2.79)\quad p(p+\nu-2)A_p^{(\nu)}(x) = 2(p+\nu)(p+\nu-2)xA_{p-1}^{(\nu)}(x) - (p+\nu)(p+2\nu-2)A_{p-2}^{(\nu)}(x)$$

$$(2.2.80)\quad \left[(p+\nu)^2-1\right]A_{p+1}^{(\nu-1)}(x) = \nu(p+\nu-1)A_{p+1}^{(\nu)}(x) - \nu(p+\nu+1)A_{p-1}^{(\nu)}(x)$$

(2.2.81) $2(\nu+1)(p+\nu)(1-x^2)A_{p-2}^{(\nu+1)}(x) = (p+\nu)(p+2\nu-1)xA_{p-1}^{(\nu)}(x)$

$$- p(p+\nu-1)A_p^{(\nu)}(x)$$

(2.2.82) $2(\nu+1)(p+\nu)(1-x^2)A_p^{(\nu+1)}(x) = (p+2\nu)(p+\nu+1)A_p^{(\nu)}(x)$

$$- (p+1)(p+\nu)xA_{p+1}^{(\nu)}(x)$$

(2.2.83) $(p+2)(p+\nu)A_{p+2}^{(\nu-1)}(x) = 2\nu(p+\nu)xA_{p+1}^{(\nu)}(x) - 2\nu(p+\nu+1)A_p^{(\nu)}(x)$

(2.2.84) $(\nu-1)[(p+\nu)^2-1]A_{p+1}^{(\nu-1)}(x) = \nu(p+\nu)(p+\nu-1)A_{p+1}^{(\nu)}(x)$

$$- 2\nu[(p+\nu)^2-1]xA_p^{(\nu)}(x) + \nu(p+\nu)(p+\nu+1)A_{p-1}^{(\nu)}(x)$$

2.2.12 Relations with Chebyshev Polynomials

(2.2.85) $A_p^{(0)}(x) = 2\ T_p(x)$

$T_p(x)$ is a Chebyshev polynomial of the first kind.

(2.2.86) $$\lim_{\nu \to 0} \frac{A_p^{(\nu)}(\cos\theta)}{A_p^{(\nu)}(1)} = T_p(\cos\theta)$$

(2.2.87) $A_p^{(1)}(x) = (p+1)\ U_p(x)$

$U_p(x)$ is a Chebyshev polynomial of the second kind.

2.2.13 Relations with Hermite Polynomials

$$(2.2.88)\quad A_p^{(\nu)}(x) = \sum_{k=0}^{[p/2]} {}_2F_0[-k,p+\nu-k;-;1]\frac{(-1)^k(p+\nu)\Gamma(p+\nu-k)}{k!\ (p-2k)!\ \Gamma(\nu+1)} H_{p-2k}(x)$$

$H_p(x)$ is a Hermite polynomial.

$$(2.2.89)\quad H_p(x) = p!\sum_{k=0}^{[p/2]} {}_1F_1[-k;p+\nu-2k+1;1]\frac{(-1)^k\Gamma(\nu+1)}{k!\ \Gamma(p+\nu-2k+1)} A_{p-2k}^{(\nu)}(x)$$

$$(2.2.90)\quad H_p(x) = p!\lim_{\nu\to\infty}\frac{A_p^{(\nu)}(x\nu^{-1/2})}{\nu^{p/2-1}(p+\nu)}$$

2.2.14 Relations with Jacobi Polynomials

$$(2.2.91)\quad A_p^{(\nu)}(x) = \frac{p!\ A_p^{(\nu)}(1)}{(\nu+1/2,p)} P_p^{(\nu-1/2,\nu-1/2)}(x)$$

$P_p^{(\alpha,\beta)}(x)$ is a Jacobi polynomial.

$$(2.2.92)\quad P_p^{(\nu,\nu)}(x) = \frac{2\ (\nu,p+1)\ A_p^{(\nu+1/2)}(x)}{p!\ (2\nu+1)\ A_p^{(\nu+1/2)}(1)}$$

$$(2.2.93)\quad A_{2p}^{(\nu)}(x) = \frac{p!\ A_p^{(\nu/2)}(1)}{(1/2,p)} P_p^{(\nu-1/2,\nu-1/2)}(2x^2-1)$$

$$(2.2.94)\quad A_{2p+1}^{(\nu)}(x) = (\nu+1)\ B(1/2,p+1)\ xA_p^{((\nu+1)/2)}(1)\ P_p^{(\nu-1/2,1/2)}(2x^2-1)$$

2.2.15 Relations with Legendre Functions

$$(2.2.95)\quad A_p^{(1/2)}(x) = (2p+1)\,P_p(x)$$

$P_p(x)$ is a Legendre polynomial.

$$(2.2.96)\quad A_p^{(\nu)}(x) = \frac{(p+\nu)}{\nu} \sum_{k=0}^{[p/2]} \frac{(\nu-1/2,k)(\nu,p-k)(1+2p-4k)}{k!\ (3/2,p-k)}\, P_{p-2k}(x)\ , \qquad \nu \neq 0$$

$$(2.2.97)\quad A_{p-k}^{(k+1/2)}(x) = \frac{(2p+1)\ 2^k\ k!}{(2k+1)\ (2k)!}\ \frac{d^k}{dx^k} P_p(x)\ , \qquad p \geq k$$

$$(2.2.98)\quad A_p^{(\nu)}(x) = \frac{(p+\nu)\ \Gamma(1/2)\ \Gamma(p+2\nu)}{p!\ \Gamma(\nu+1)} \left[4(x^2-1)\right]^{1/4-\nu/2} P_{p+\nu-1/2}^{1/2-\nu}(x)$$

$P_p^q(x)$ is an associated Legendre function of the first kind.

2.2.16 Rodrigue's Formula

$$(2.2.99)\quad A_p^{(\nu)}(x) = \frac{(-1)^p\ A_p^{(\nu)}(1)}{2^p\ (\nu+1/2,p)\ w^{(\nu)}(x)}\ D^p\left[(1-x^2)^p\ w^{(\nu)}(x)\right]$$

$w^{(\nu)}(x)$ is as defined in (2.2.62).

2.2.17 Series and Expansions Involving Gegenbauer Polynomials

$$(2.2.100)\quad x^p = \frac{p!}{2^p} \sum_{k=0}^{[p/2]} \frac{A_{p-2k}^{(\nu)}(x)}{k!\ (\nu+1,p-k)}\ , \qquad p \geq 0,\ -1 \leq x \leq 1$$

See Table 2.4 for explicit formulas for $1 \leq p \leq 10$.

$$\exp(xt) = (t/2)^{-\nu}\,\Gamma(\nu+1)\sum_{p=0}^{\infty} I_{p+\nu}(t)\,A_p^{(\nu)}(x) \tag{2.2.101}$$

$$\nu > 0,\ -1 < x < 1$$

$$\operatorname{sgn}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\pi}\,B(k+1/2,\ +1)\,A_{2k+1}^{(\nu)}(x)\ , \tag{2.2.102}$$

$$\nu > -1/2,\ -1 < x < 1$$

$$|x| = \frac{1}{\nu+1} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\pi(2k-1)}\,B(k+1/2,\nu+1)\,A_{2k}^{(\nu)}(x)\ , \tag{2.2.103}$$

$$\nu > -1/2,\ -1 < x < 1$$

$$(1-x)^p = 2^p p!\,\Gamma(2\nu+1)\,(\nu+1/2,p)\sum_{k=0}^{p} \frac{(-1)^k\,A_k^{(\nu)}(x)}{(p-k)!\,\Gamma(p+2\nu+k+1)}\ , \tag{2.2.104}$$

$$-1 < x < 1,\ -p < (\nu+1)/2 \text{ if } \nu > 0,\ -p < \nu+1/2 \text{ if } -1/2 < \nu < 0$$

$$\|\underline{x}-\underline{y}\|^{-2\nu} = \sum_{p=0}^{\infty} \frac{\nu}{p+\nu}\,\frac{\|\underline{y}\|^p}{\|\underline{x}\|^{p+2\nu}}\,A_p^{(\nu)}(\cos\theta)\ , \tag{2.2.105}$$

$$\underline{x},\underline{y} \in E^m,\ \|\underline{x}\| > \|\underline{y}\|,\ \cos\theta = \underline{x}'\underline{y}/(\|\underline{x}\|\,\|\underline{y}\|)$$

$$\left(\frac{x}{4\sin\frac{\theta}{2}}\right)^{\nu} \frac{J_\nu(2x\sin\frac{\theta}{2})}{\Gamma(\nu+1)} = \sum_{p=0}^{\infty} \left[J_{p+\nu}(x)\right]^2 A_p^{(\nu)}(\cos\theta) \tag{2.2.106}$$

$$A_p^{(\nu)}\left(-\frac{1-\lambda x}{r}\right) = p!\,A_p^{(\nu)}(1)\sum_{k=0}^{\infty} \frac{(-1)^k\,\lambda^k}{k!\,(p-k)!}\,\frac{A_k^{(\nu)}(x)}{A_k^{(\nu)}(1)}\ , \tag{2.2.107}$$

$$r = (1-2\lambda x+\lambda^2)^{1/2},\ |\lambda| < 1$$

$$(2.2.108)\quad A_p^{(\nu)}\left(\frac{x-\lambda}{r}\right) = r^{p+2\nu}\frac{(p+\nu)}{p}\sum_{k=0}^{\infty}\frac{A_{p+k}^{(\nu)}(x)\ \lambda^k}{(p+\nu+k)\ B(p,k+1)},$$

$$r = (1-2\lambda x+\lambda^2)^{1/2}$$

$$(2.2.109)\quad A_p^{(\nu)}(\cos\theta) = (p+\nu)\left(\frac{\sin\theta}{\sin\phi}\right)^p \sum_{k=0}^{p}\frac{(2\nu+k,p-k)}{(p-k)!\ (k+\nu)} \times \left(\frac{\sin(\phi-\theta)}{\sin\theta}\right)^{p-k} A_k^{(\nu)}(\cos\phi)$$

$$\sin\theta\ \sin\phi \neq 0$$

$$(2.2.110)\quad A_p^{(\nu)}(\cos 2\theta) = \nu(p+\nu)\sum_{k=0}^{2p}\frac{(-1)^k A_k^{(\nu)}(\cos\theta)A_{2p-k}^{(\nu)}(\cos\theta)}{(k+\nu)\ (2p-k)},$$

$$(2.2.111)\quad A_p^{(\nu/2)}(\cos(\theta-\phi)) = \frac{A_p^{(\nu/2-1/2)}(1)}{\nu(2p+\nu-1,2)\ B(\nu/2,\nu/2)} \times \sum_{k=0}^{p}\frac{(p-k)!\ (\nu+2k)\ (\nu+2k-1,2)\ (4\sin\theta\ \sin\phi)^k}{\Gamma(p+\nu+k)} \times A_{p-k}^{(\nu/2+k)}(\cos\theta)\ A_{p-k}^{(\nu/2+k)}(\cos\phi)$$

$$(2.2.112)\quad \left(\frac{\alpha}{2}\right)^{1/2}(\sin\theta\ \sin\phi)^{1/2-\nu} J_{\nu-1/2}(\alpha\ \sin\theta\ \sin\phi)\ \exp(\imath\alpha\ \cos\theta\ \cos\phi) = \nu 2^{\sum_{k=0}^{\infty}}\frac{\imath^k k!\ J_{k+\nu}(\alpha)}{(k+\nu)(\nu,2k)(2\nu,k)}\ A_k^{(\nu)}(\cos\theta)\ A_k^{(\nu)}(\cos\phi),$$

$$\nu > 0,\ 0 < \theta < \pi,\ 0 < \phi < \pi$$

Table 2.4 Coefficients b_s for x^p in terms of $A_s^{(\nu)}(x)$ on S_m, $1 \leq p \leq 10$

$$x^p = \sum_{s=0}^{p} b_s A_s^{(\nu)}(x) \qquad (\nu + 1)_n = (\nu + 1)(\nu + 2)\cdots(\nu + n)$$

p	$A_0^{(\nu)}(x)$	$A_1^{(\nu)}(x)$	$A_2^{(\nu)}(x)$	$A_3^{(\nu)}(x)$	$A_4^{(\nu)}(x)$	$A_5^{(\nu)}(x)$	$A_6^{(\nu)}(x)$	$A_7^{(\nu)}(x)$	$A_8^{(\nu)}(x)$	$A_9^{(\nu)}(x)$	$A_{10}^{(\nu)}(x)$
1		$\frac{1}{2(\nu+1)}$									
2	$\frac{1}{2(\nu+1)}$		$\frac{1}{2(\nu+1)_2}$								
3		$\frac{3}{4(\nu+1)_2}$		$\frac{3}{4(\nu+1)_3}$							
4	$\frac{3}{4(\nu+1)_2}$		$\frac{3}{2(\nu+1)_3}$		$\frac{3}{2(\nu+1)_4}$						
5		$\frac{15}{8(\nu+1)_3}$		$\frac{15}{4(\nu+1)_4}$		$\frac{15}{4(\nu+1)_5}$					
6	$\frac{15}{8(\nu+1)_3}$		$\frac{45}{8(\nu+1)_4}$		$\frac{45}{4(\nu+1)_5}$		$\frac{45}{4(\nu+1)_6}$				
7		$\frac{105}{16(\nu+1)_4}$		$\frac{315}{16(\nu+1)_5}$		$\frac{315}{8(\nu+1)_6}$		$\frac{315}{8(\nu+1)_7}$			
8	$\frac{105}{16(\nu+1)_4}$		$\frac{105}{4(\nu+1)_5}$		$\frac{315}{4(\nu+1)_6}$		$\frac{315}{2(\nu+1)_7}$		$\frac{315}{2(\nu+1)_8}$		
9		$\frac{945}{32(\nu+1)_5}$		$\frac{945}{8(\nu+1)_6}$		$\frac{2835}{8(\nu+1)_7}$		$\frac{2835}{4(\nu+1)_8}$		$\frac{2835}{4(\nu+1)_9}$	
10	$\frac{945}{32(\nu+1)_5}$		$\frac{4725}{32(\nu+1)_6}$		$\frac{4725}{8(\nu+1)_7}$		$\frac{14175}{8(\nu+1)_8}$		$\frac{14175}{4(\nu+1)_9}$		$\frac{14175}{4(\nu+1)_{10}}$

2.3 Density Representation on the Hypersphere

In section 1.4 densities on S_m of practical interest were noted to be of the form $f(\lambda \underline{u}'\underline{\xi})$ with mode $\underline{\xi}$ and rotational symmetry about $\underline{\xi}$. Here we will present representations of such densities in terms of spherical harmonic functions on S_m.

As a starting point, we will motivate our choice of density decomposition of S_m by considering the exponential family of densities of S_m. Watson (1983) gives the exponential family to be

$$C \exp\{\sum_{p=1}^{k} G_p(\lambda \underline{u}'\underline{\xi})\} \qquad \underline{u},\underline{\xi} \in S_m$$

where $G_p(\lambda \underline{u}'\underline{\xi})$ is a homogeneous polynomial of degree p and C is a normalizing constant given by

$$C^{-1} = \int_{S_m} \exp\{\sum_{p=1}^{k} G_p(\lambda \underline{u}'\underline{\xi})\}\, dU^{(\nu)}(\underline{u}).$$

Following Watson (1983), the constant C will, in general be a complicated function of the coefficients of the polynomials $\{G_p(\lambda \underline{u}'\underline{\xi})\}_{p=1}^{k}$. Hence it would be desirable to find a simple representation of the linear space of homogeneous polynomials of degree p. Further, we require the representation to be invariant under rotations of the coordinate system which leave $\underline{\xi}$ fixed. That is, if $\underline{w} = P\underline{u}$ where P is an orthogonal matrix which leaves $\underline{\xi}$ invariant, then $\underline{w}'\underline{\xi} = (P\underline{u})'\underline{\xi} = (P\underline{u})'(P\underline{\xi}) = \underline{u}'\underline{\xi}$ and thus $G_p(\lambda \underline{w}'\underline{\xi}) = G_p(\lambda \underline{u}'\underline{\xi})$.

Now consider $S_p(m,\underline{u})$, $\underline{u} \in S_m$, a spherical harmonic function of degree p in m dimensions as introduced in section 2.1. It is well known

(see Müller (1966)) that there are $(2p+m-2)(p+m-3)! \,/\, [(m-2)!p!] = A_p^{(\nu)}(1)$, $\nu = m/2 - 1$, linear independent spherical harmonics of degree p in m dimensions which (1) form a basis for the linear space of homogeneous polynomials of degree p, (2) are closed with respect to rotations of the coordinates by an orthogonal matrix and (3) are orthogonal over S_m in the sense that

$$\int_{S_m} S_i(m,\underline{u})\, S_j(m,\underline{u})\, dU^{(\nu)}(\underline{u}) = 0$$

for $1 \leq i,j \leq A_p^{(\nu)}(1)$ and $i \neq j$.

For densities restricted to have rotational symmetry about some vector $\underline{\xi} \in S_m$ we have that if P is an orthogonal matrix such that $(P\underline{u})'\underline{\xi} = \underline{u}'\underline{\xi}$ then $S_p(m,P\underline{u}) = S_p(m,\underline{u})$. By standardizing $S_p(m,1)$ to be unity, Müller (1966) writes $S_p(m,\underline{u})$ as $P_p(m,\underline{u}'\underline{\xi})$, a Legendre polynomial of degree p in m dimensions where $P_p(m,\underline{u}'\underline{\xi})$ is uniquely determined.

On the circle, S_2, $P_p(2,\underline{u}'\underline{\xi})$ may be expressed in terms of the Chebyshev polynomials $T_p(\underline{u}'\underline{\xi})$, or equivalently by (2.2.85), the Gegenbauer polynomials $A_p^{(0)}(\underline{u}'\underline{\xi})$. On the sphere, S_3, $P_p(3,\underline{u}'\underline{\xi})$ is the usual Legendre polynomial $P_p(\underline{u}'\underline{\xi})$, which by (2.2.95) may be expressed in terms of the Gegenbauer polynomials $A_p^{(1/2)}(\underline{u}'\underline{\xi})$. In general, for $t = \underline{u}'\underline{\xi} \in [-1,1]$, $P_p(m,t)$ satisfies

$$\left[(1-t^2)D^2 - (2\nu+1)tD + p(p+2\nu)\right] P_p(m,t) = 0$$

which by (2.2.5) shows that $P_p(m,\underline{u}'\underline{\xi})$ may be written in terms of $A_p^{(\nu)}(\underline{u}'\underline{\xi})$. In fact, $A_p^{(\nu)}(t) = A_p^{(\nu)}(1)\, P_p(2(\nu+1),t)$ with $t \in [-1,1]$.

Thus the Gegenbauer polynomials $\{A_p^{(\nu)}(\underline{u}'\underline{\xi})\}$ may serve to represent

rotationally symmetric densities on S_m which are functions of homogeneous polynomials. Since it is known that the Gegenbauer polynomials $\{A_p^{(\nu)}(x)\}$ are complete (with respect to $U^{(\nu)}$ on S_m) for the subset of continuous functions on S_m which are of the form $f(\underline{u}'\underline{\xi})$, we have the following:

Definition 2.3.1: Let $g(\underline{u}'\underline{\xi})$ be a function on S_m. Then

$$\sum_{p=0}^{\infty} \bar{g}_p^{(\nu)} A_p^{(\nu)}(\underline{u}'\underline{\xi}), \qquad \nu = \frac{m}{2} - 1,$$

will be called the formal Gegenbauer transform of $g(\underline{u}'\underline{\xi})$ where

$$\bar{g}_p^{(\nu)} = [A_p^{(\nu)}(1)]^{-1} \int_{S_m} g(\underline{u}'\underline{\xi})\, A_p^{(\nu)}(\underline{u}'\underline{\xi})\, dU^{(\nu)}(\underline{u}).$$

Provided that

$$\lim_{n\to\infty} \int_{S_m} \left[g(\underline{u}'\underline{\xi}) - \sum_{p=0}^{n} \bar{g}_p^{(\nu)} A_p^{(\nu)}(\underline{u}'\underline{\xi})\right]^2 dU^{(\nu)}(\underline{u}) = 0,$$

then we will call

$$\sum_{p=0}^{\infty} \bar{g}_p^{(\nu)} A_p^{(\nu)}(\underline{u}'\underline{\xi})$$

the Gegenbauer transform of $g(\underline{u}'\underline{\xi})$.

According to Erdélyi (1953), if $g(x)$ is a continuous function of the real variable x with $-1 \leq x \leq 1$, then the Gegenbauer transform of $g(x)$ exists. Pointwise convergence of $\sum_p \bar{g}_p^{(\nu)} A_p^{(\nu)}(\underline{u}'\underline{\xi})$ to $g(\underline{u}'\underline{\xi})$ almost everywhere with respect to $U^{(\nu)}$ requires

$$\sum_{p=0}^{\infty} [\bar{g}_p^{(\nu)}]^2 A_p^{(\nu)}(1) < \infty .$$

We now present a table of the coefficients $\bar{g}_p^{(\nu)}$ for several well known densities on S_m (refer to Saw (1984)). For the Gegenbauer transform of other functions see the listing of formulas in subsection 2.2.17.

Table 2.5 Coefficients, $\bar{g}_p^{(\nu)}$, of Gegenbauer Transforms of Densities, $g(\lambda \underline{u}'\underline{\xi})$, on S_m

Distribution	$g(\lambda \underline{u}'\underline{\xi})$	$\bar{g}_p^{(\nu)}$, $p = 0,1,2,\ldots$
Langevin	$\Lambda_\nu^{-1}(\lambda)\exp(\lambda \underline{u}'\underline{\xi})$ $\Lambda_\nu(\lambda) = {}_0F_1(-;\nu+1;\lambda^2/4)$	$I_{p+\nu}(\lambda)/I_\nu(\lambda)$
Saw's	$e^{-\lambda^2}\sum_{k=0}^{\infty}\frac{(2\lambda \underline{u}'\underline{\xi})^k(\nu+1,k/2)}{k!}$	$\frac{\lambda^p N(p/2,p+\nu+1;-\lambda^2)}{(\nu+1+p/2,p/2)}$
Saw's Hyperspherical t	$\frac{K_s^{(\nu)}(\lambda)}{(1-2\lambda \underline{u}'\underline{\xi}+\lambda^2)^{\nu+s}}$ $[K_s^{(\nu)}(\lambda)]^{-1} = {}_2F_1(s,\nu+s;\nu+1;\lambda^2)$	$\frac{\lambda^p(\nu+s,p)\,K_s^{(\nu)}(\lambda)}{(\nu+1,p)\,K_s^{(\nu+p)}(\lambda)}$
Cauchy	$\frac{1-\lambda^2}{(1-2\lambda \underline{u}'\underline{\xi}+\lambda^2)^{\nu+1}}$	λ^p
Cardioid	$1 + \lambda A_1^{(\nu)}(\underline{u}'\underline{\xi})$	1 if $p = 0$ λ if $p = 1$ 0 if $p \geq 2$
Wrapped Normal	$\sum_{p=0}^{\infty}\lambda^{p^2}A_p^{(\nu)}(\underline{u}'\underline{\xi})$	λ^{p^2}
Point	$\sum_{p=0}^{\infty}A_p^{(\nu)}(\underline{u}'\underline{\xi})$	1

CHAPTER 3
THE STATISTIC $Y_p(\cdot)$

3.1 Introduction

In section 1.6 tests of uniformity proposed by Ajne (1968) were discussed. Ajne's tests were developed for testing uniformity on the unit circle S_2. Of central importance in those tests is the statistic $N(\pi,x)$ which counts the number of sample points in some semicircle corresponding to the interval $(x,x+\pi]$.

The analogue of the statistic $N(\pi,x)$ on m-dimensional spheres should count the number of sample points on some m-dimensional hemisphere. We denote the m-dimensional hemisphere $H_m(\underline{u})$, rotationally symmetric about $\underline{u} \in S_m$, by

$$H_m(\underline{u}) = \{\underline{x} \in S_m : \underline{x}'\underline{u} > 0\}.$$

Then given the random sample $\underline{x}_1,\ldots,\underline{x}_n$ from S_m, define $N(\underline{u})$ to be the number of sample points in $H_m(\underline{u})$. In terms of indicator functions,

$$N(\underline{u}) = \sum_{j=1}^{n} I_{\{H_m(\underline{u})\}}(\underline{x}_j).$$

Under the hypothesis of uniformity, $N(\underline{u})$ follows a binomial distribution with parameters n and $p = 1/2$.

Generalizations of Ajne's results to m-dimensional spheres have been recounted in section 1.6. However, the test statistics in those generalizations have complicated asymptotic distributions. In section 3.3 we shall propose a test statistic, $Y_p(\cdot)$, motivated by $N(\underline{u})$, which has a normal asymptotic distribution under the hypothesis of uniformity.

3.2 The Functions $X(\cdot)$ and $\delta(\cdot)$

Under the null hypothesis of uniformity we standardize $N(\underline{u})$, the number of sample points $\underline{x}_1,\ldots,\underline{x}_n$ which occur in $H_m(\underline{u})$, and define

$$X(\underline{u}) = 2n^{-1/2}\ (N(\underline{u}) - n/2).$$

Both Ajne's statistic I and Prentice's m-dimensional analogue $A_{m,n}$, discussed in section 1.6, are functions of $X(\underline{u})$.

In future work we will find alternate expressions for $X(\underline{u})$ useful. As a first step toward deriving alternate expressions for $X(\underline{u})$, notice that to count the number of sample points in $H_m(\underline{u})$ is equivalent to subtracting the number of sample points not in $H_m(\underline{u})$ from the total sample size n. Supposing that no sample points fall on the boundary between $H_m(\underline{u})$ and $S_m - H_m(\underline{u}) = H_m(-\underline{u})$, then those sample points not in $H_m(\underline{u})$ belong to $H_m(-\underline{u})$ and hence $N(\underline{u}) = n - N(-\underline{u})$. Thus we may write

$$X(\underline{u}) = n^{-1/2}\ (2N(\underline{u}) - n)$$

$$= n^{-1/2}\ [N(\underline{u}) - (n - N(\underline{u}))]$$

$$= n^{-1/2}\ (N(\underline{u}) - N(-\underline{u})).$$

Observe that $N(\underline{u}) - N(-\underline{u})$ counts the difference between those sample points in $H_m(\underline{u})$ and those in $H_m(-\underline{u})$.

Now for $\underline{u} \in S_m$ and $\underline{\xi} \in S_m$ define the signum function $\operatorname{sgn}(\underline{u}'\underline{\xi}) = \delta(\underline{u}'\underline{\xi})$ as

$$\delta(\underline{u}'\underline{\xi}) = \begin{cases} 1 & \text{if } \underline{u}'\underline{\xi} > 0, \\ 0 & \text{if } \underline{u}'\underline{\xi} = 0, \\ -1 & \text{if } \underline{u}'\underline{\xi} < 0. \end{cases} \tag{3.2.1}$$

Thus $\delta(\underline{u}'\underline{x}) = 1$ if $\underline{x} \in H_m(\underline{u})$ and $\delta(\underline{u}'\underline{x}) = -1$ if $\underline{x} \in H_m(-\underline{u})$. We may now write

$$N(\underline{u}) - N(-\underline{u}) = \sum_{j=1}^{n} \delta(\underline{u}'\underline{x}_j)$$

so that

$$X(\underline{u}) = n^{-1/2} \sum_{j=1}^{n} \delta(\underline{u}'\underline{x}_j). \tag{3.2.2}$$

Since $N(\underline{u}) \sim$ Binomial $(n,1/2)$ under the null hypothesis of uniformity, it is obvious that

$$EX(\underline{u}) = 0 \quad \text{and} \quad \mathrm{Var}(X(\underline{u})) = 1.$$

To compute $\mathrm{Cov}(X(\underline{u}),X(\underline{w}))$ we first prove the following:

Theorem 3.2.1: Let $\underline{x} \sim U(S_m)$ and $\underline{u},\underline{w} \in S_m$. Then

$$E\delta(\underline{x}'\underline{u}) = 0,$$

$$\mathrm{Cov}(\delta(\underline{x}'\underline{u}),\delta(\underline{x}'\underline{w})) = (2/\pi)\sin^{-1}(\underline{u}'\underline{w}).$$

Proof: Since $\underline{x} \sim U(S_m)$ we have

$$\begin{aligned} E\delta(\underline{x}'\underline{u}) &= P(\underline{x} \in H_m(\underline{u})) - P(\underline{x} \in H_m(-\underline{u})) \\ &= 1/2 - 1/2 \\ &= 0. \end{aligned}$$

To compute the covariance we first divide S_m into the following four regions:

$$R_1 = H_m(\underline{u}) \cap H_m(\underline{w}) = \{\underline{x} \in S_m : \underline{x}'\underline{u} > 0, \ \underline{x}'\underline{w} > 0\},$$

$$R_2 = H_m(\underline{u}) \cap H_m(-\underline{w}) = \{\underline{x} \in S_m : \underline{x}'\underline{u} > 0, \ \underline{x}'\underline{w} < 0\},$$

$$R_3 = H_m(-\underline{u}) \cap H_m(\underline{w}) = \{\underline{x} \in S_m : \underline{x}'\underline{u} < 0, \ \underline{x}'\underline{w} > 0\},$$

$$R_4 = H_m(-\underline{u}) \cap H_m(-\underline{w}) = \{\underline{x} \in S_m : \underline{x}'\underline{u} < 0, \ \underline{x}'\underline{w} < 0\}.$$

These four regions are illustrated in Figure 3.1 for the case m = 2.

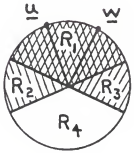

Figure 3.1 Regions R_1, R_2, R_3 and R_4 for m = 2

Let $p_i = P(\underline{x} \in R_i)$, $i = 1,\ldots,4$, $\underline{x} \sim U(S_m)$. It is sufficient to only compute p_1 because for $\underline{x} \sim U(S_m)$, $p_1 = p_4$ and $p_2 = p_3 = 1/2 - p_1$. Then using Theorem 1.3.1 to write $\underline{x} \sim U(S_m)$ as $\underline{x} = \underline{z} / \|\underline{z}\|$ where $\underline{z} \sim N_m(0, I_m)$ we have

$$p_1 = P(\underline{x} \in R_1)$$

$$= P(\underline{x}'\underline{u} > 0, \ \underline{x}'\underline{w} > 0)$$

$$= P\left(\frac{\underline{z}'\underline{u}}{\|\underline{z}\|} > 0, \ \frac{\underline{z}'\underline{w}}{\|\underline{z}\|} > 0\right)$$

$$= P(\underline{z}'\underline{u} > 0,\ \underline{z}'\underline{w} > 0)$$

$$= P(y_1 > 0,\ y_2 > 0)$$

where $y_1 = \underline{z}'\underline{u}$ and $y_2 = \underline{z}'\underline{w}$. But

$$y_1 = \underline{z}'\underline{u} \sim N\Big(0,\ \sum_{i=1}^{m} u_i^2 = 1\Big) ,$$

and

$$y_2 = \underline{z}'\underline{w} \sim N\Big(0,\ \sum_{i=1}^{m} w_i^2 = 1\Big) .$$

Further,

$$\mathrm{Cov}(y_1,y_2) = \mathrm{Cov}(\underline{z}'\underline{u},\ \underline{z}'\underline{w})$$

$$= \mathrm{Cov}\Big(\sum_{i=1}^{m} u_i z_i,\ \sum_{j=1}^{m} w_j z_j\Big)$$

$$= \sum_{i=1}^{m} u_i w_i\ \mathrm{Cov}(z_i,z_i)$$

$$= \underline{u}'\underline{w}.$$

Thus,

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \sim N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix},\ \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right)$$

where $\rho = \underline{u}'\underline{w}$ gives

$$P_1 = \frac{1}{2\pi(1-\rho^2)^{1/2}} \int_0^{\infty}\int_0^{\infty} \exp\Big\{\frac{1}{2(1-\rho^2)}\ (y_1^2 - 2\rho y_1 y_2 + y_2^2)\Big\}\ dy_1 dy_2 .$$

Now let $y_1 = s(1-\rho^2)^{1/2} + \rho t$ and $y_2 = t$. Since the Jacobian of the transformation from $\{y_1, y_2\}$ to $\{s,t\}$ is $J\{y_1, y_2 \to s,t\} = (1-\rho^2)^{1/2}$, then

$$p_1 = \frac{1}{2\pi} \int_0^{\infty} \int_{\frac{-\rho}{(1-\rho^2)^{1/2}}}^{\infty} \exp\left\{ \frac{-1}{2} (s^2 + t^2) \right\} ds dt.$$

Changing to polar coordinates with $r^2 = s^2 + t^2$, $s = r \cos\theta$, $t = r \sin\theta$ and $J\{s,t \to r,\theta\} = r$, we have

$$= \frac{1}{2\pi} \int_{\tan^{-1}\frac{-\rho}{(1-\rho^2)^{1/2}}}^{\pi/2} \int_0^{\infty} r \exp\{-r^2/2\} \, dr d\theta$$

$$= \frac{1}{2\pi} \left\{ \frac{\pi}{2} - \tan^{-1} \frac{-\rho}{(1-\rho^2)^{1/2}} \right\}$$

$$= \frac{1}{4} + \frac{1}{2\pi} \tan^{-1} \frac{-\rho}{(1-\rho^2)^{1/2}}$$

$$= \frac{1}{4} + \frac{1}{2\pi} \left\{ \frac{\pi}{2} - \cos^{-1}\rho \right\}$$

$$= \frac{1}{2} - \frac{1}{2\pi} \cos^{-1}(\underline{u}'\underline{w}).$$

Observe that,

$$\delta(\underline{x}'\underline{u}) \; \delta(\underline{x}'\underline{w}) = \begin{cases} 1 & \text{if } \underline{x} \in R_1 \text{ or } \underline{x} \in R_4, \\ -1 & \text{if } \underline{x} \in R_2 \text{ or } \underline{x} \in R_3. \end{cases}$$

Hence,

$$\begin{aligned}
\mathrm{Cov}(\delta(\underline{x}'\underline{u})\ \delta(\underline{x}'\underline{w})) &= E(\delta(\underline{x}'\underline{u})\ \delta(\underline{x}'\underline{w})) \\
&= P(\underline{x} \in R_1 \text{ or } \underline{x} \in R_4) - P(\underline{x} \in R_2 \text{ or } \underline{x} \in R_3) \\
&= p_1 + p_4 - p_2 - p_3 \\
&= 4p_1 - 1 \\
&= 4\left\{\frac{1}{2} - \frac{1}{2\pi}\cos^{-1}(\underline{u}'\underline{w})\right\} - 1 \\
&= 1 - \frac{2}{\pi}\cos^{-1}(\underline{u}'\underline{w}) \\
&= 1 - \frac{2}{\pi}\left\{\frac{\pi}{2} - \sin^{-1}(\underline{u}'\underline{w})\right\} \\
&= \frac{2}{\pi}\sin^{-1}(\underline{u}'\underline{w}).
\end{aligned}$$

With Theorem 3.2.1 we may prove the following:

Theorem 3.2.2: Let $\underline{x}_j \sim$ i.i.d. $U(S_m)$, $j = 1,\ldots,n$ and $X(\underline{u}) = n^{-1/2}(2N(\underline{u}) - n)$, $\underline{u} \in S_m$ where $N(\underline{u}) = \sum_{j=1}^{n}(1 + \delta(\underline{u}'\underline{x}_j))/2$. Then

$$EX(\underline{u}) = 0$$

$$\mathrm{Cov}(X(\underline{u}),X(\underline{w})) = \frac{2}{\pi}\sin^{-1}(\underline{u}'\underline{w}).$$

Proof: It has already been established that $EX(\underline{u}) = 0$. To compute the covariance, we express $X(\underline{u})$ in terms of the signum function and use Theorem 3.2.1 to find

$$\mathrm{Cov}(X(\underline{u}),X(\underline{w})) = \mathrm{Cov}\Big(n^{-1/2}\sum_{i=1}^{n}\delta(\underline{u}'\underline{x}_i),\ n^{-1/2}\sum_{j=1}^{n}\delta(\underline{w}'\underline{x}_j)\Big)$$

$$= n^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n} E\big(\delta(\underline{u}'\underline{x}_i)\ \delta(\underline{w}'\underline{x}_j)\big)$$

$$= n^{-1}\sum_{i=1}^{n} E\big(\delta(\underline{u}'\underline{x}_i)\ \delta(\underline{w}'\underline{x}_j)\big)$$

$$+ n^{-1}\sum_{i\neq j}\sum E\big(\delta(\underline{u}'\underline{x}_i)\big)\ E\big(\delta(\underline{w}'\underline{x}_j)\big)$$

$$= n^{-1}\sum_{i=1}^{n}\frac{2}{\pi}\sin^{-1}(\underline{u}'\underline{w})$$

$$= \frac{2}{\pi}\sin^{-1}(\underline{u}'\underline{w}).$$

Returning to the signum function $\delta(\underline{u}'\underline{\xi})$, we verify formula (2.2.102) that the Gegenbauer transform of $\delta(\underline{u}'\underline{\xi})$ is

$$\sum_{p=1}^{\infty}\bar{\delta}_p\, A_p^{(\nu)}(\underline{u}'\underline{\xi})$$

where

$$\bar{\delta}_p = \begin{cases} \dfrac{(-1)^{(p-1)/2}}{\pi}\, B(p/2,\nu+1) & p = 1,3,5,\ldots, \\ 0 & p = 0,2,4,\ldots. \end{cases}$$

By definition 2.3.1 and formula (2.2.67),

$$\int_{S_m} \delta(\underline{u}'\underline{\xi})\, A_p^{(\nu)}(\underline{u}'\underline{\xi})\, dU^{(\nu)}(\underline{u})$$

$$= \int_{S_m} \{ \sum_{q=0}^{\infty} \bar{\delta}_q\, A_q^{(\nu)}(\underline{u}'\underline{\xi}) \}\, A_p^{(\nu)}(\underline{u}'\underline{\xi})\, dU^{(\nu)}(\underline{u})$$

$$= \bar{\delta}_p \int_{S_m} \{A_p^{(\nu)}(\underline{u}'\underline{\xi})\}^2\, dU^{(\nu)}(\underline{u})$$

$$= \bar{\delta}_p\, A_p^{(\nu)}(1).$$

Since $A_p^{(\nu)}(x)$ is an odd (even) function for p odd (even) and $\delta(x)$ is an odd function, we have

$$\bar{\delta}_p\, A_p^{(\nu)}(1) = \begin{cases} 2 \int_0^1 A_p^{(\nu)}(x)\, dW^{(\nu)}(x) & p = 1,3,5,\ldots, \\ 0 & p = 0,2,4,\ldots. \end{cases}$$

Then for p = 1,3,5,..., formula (2.2.63) gives

$$\bar{\delta}_p = \frac{4(\nu+1)\, A_{p-1}^{(\nu+1)}(0)}{p(p+2\nu)\, B(1/2, \nu+1/2)\, A_p^{(\nu)}(1)}.$$

Upon simplification using formulas (2.2.31) and (2.2.32) we get

$$\bar{\delta}_p = \frac{(-1)^{(p-1)/2}}{\pi} B(p/2, \nu+1), \qquad p = 1,3,5,\ldots.$$

We now turn to the Gegenbauer expansion of $\mathrm{Cov}(X(\underline{\xi}), X(\underline{\eta})) = 2/\pi \sin^{-1}(\underline{\xi}'\underline{\eta}) = \tau(w)$, say, where $w = \underline{\xi}'\underline{\eta}$, $\underline{\xi}, \underline{\eta} \in S_m$. We note in passing, that $\tau(w^{1/2})$ is the distribution function of the well known arcsine distribution.

Theorem 3.2.3: The coefficients of the Gegenbauer transform of $\tau(w) = 2/\pi \sin^{-1} w$ are given by

$$\bar{\tau}_p = \bar{\delta}_p^2 \qquad p = 0,1,2,\ldots$$

where $\bar{\delta}_p$, $p = 0,1,2,\ldots$ are the coefficients of the Gegenbauer transform of the signum function $\delta(w)$.

Proof: Let $w = \underline{\xi}'\underline{\eta}$, with $\underline{\xi},\underline{\eta} \in S_m$. Then

$$\tau(w) = \tau(\underline{\xi}'\underline{\eta})$$

$$= \mathrm{Cov}(\delta(\underline{u}'\underline{\xi}),\delta(\underline{u}'\underline{\eta})), \qquad \underline{u} \sim U(S_m)$$

$$= E\{[\sum_{p=0}^{\infty} \bar{\delta}_p A_p^{(\nu)}(\underline{u}'\underline{\xi})][\sum_{q=0}^{\infty} \bar{\delta}_q A_q^{(\nu)}(\underline{u}'\underline{\eta})]\}$$

$$= E\{\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \bar{\delta}_p \bar{\delta}_q A_p^{(\nu)}(\underline{u}'\underline{\xi}) A_q^{(\nu)}(\underline{u}'\underline{\eta})\}$$

$$= \sum_{p=0}^{\infty} \bar{\delta}_p^2 A_p^{(\nu)}(\underline{\xi}'\underline{\eta}), \qquad \text{by (2.2.67)}$$

$$= \sum_{p=0}^{\infty} \bar{\tau}_p A_p^{(\nu)}(\underline{\xi}'\underline{\eta}).$$

Thus by the uniqueness of the coefficients, we have $\bar{\tau}_p = \bar{\delta}_p^2$, $p = 0,1,2,\ldots$.

3.3 The Statistic $Y_p(\cdot)$ under the Hypothesis of Uniformity

In this section we shall introduce the statistic $Y_p(\cdot)$ and note several properties of $Y_p(\cdot)$ under the hypothesis of uniformity. Section 3.4 will examine $Y_p(\cdot)$ under various alternative hypotheses while Chapter 4 will discuss testing uniformity with $Y_p(\cdot)$.

The generalization, $A_{m,n}$, of Ajne's statistic $I/(2\pi n)$ was given by formula (1.6.3) and may be written in terms of the measure $U^{(\nu)}$ as

$$A_{m,n} = n^{-1} \int_{S_m} \{N(\underline{u}) - n/2\}^2 \, dU^{(\nu)}(\underline{u})$$

$$= 1/4 \int_{S_m} \{X(\underline{u})\}^2 \, dU^{(\nu)}(\underline{u}).$$

However, it is somewhat difficult to work with the asymptotic null distribution of $A_{m,n}$ which involves an infinite series of weighted chi-square random variables. We are thereby motivated to find a kernel for the integral of $\{N(\underline{\xi}) - n/2\}$ which simplifies the asymptotic distribution. The relation between this new integral and $A_{m,n}$ is indicated in section 4.1. Now we make the following:

Definition 3.3.1: Let $\underline{x}_1, \ldots, \underline{x}_n$ be a random sample from S_m. Set $X(\underline{u}) = n^{-1/2}(2N(\underline{u}) - n)$ where $\underline{u} \in S_m$, $N(\underline{u}) = \sum_j (1 + \delta(\underline{u}'\underline{x}_j))/2$ and $\delta(w)$ is the signum function. Then for $p = 0,1,2,\ldots$ and $\underline{\xi} \in S_m$, define

$$Y_p(\underline{\xi}) = (-1)^{(p-1)/2} \int_{S_m} X(\underline{u}) \, A_p^{(\nu)}(\underline{u}'\underline{\xi}) \, dU^{(\nu)}(\underline{u}).$$

A computational form for $Y_p(\underline{\xi})$ will now be derived by using the expression (3.2.2) for $X(\underline{u})$ written in terms of the signum function. In

addition, the computational form of $Y_p(\underline{\xi})$ will show that

$$Y_p(\underline{\xi}) = 0 \text{ for } p = 0,2,4,\ldots.$$

Substituting $X(\underline{u}) = n^{-1/2} \sum_j \delta(\underline{u}'\underline{x}_j)$ and the Gegenbauer transform of $\delta(w)$ into the definition of $Y_p(\underline{\xi})$ yields

$$Y_p(\underline{\xi}) = (-1)^{(p-1)/2} \int_{S_m} n^{-1/2} \sum_{j=1}^{n} \delta(\underline{u}'\underline{x}_j)\, A_p^{(\nu)}(\underline{u}'\underline{\xi})\, dU^{(\nu)}(\underline{u})$$

$$= (-1)^{(p-1)/2}\, n^{-1/2} \sum_{j=1}^{n} \int_{S_m} \Big(\sum_{q=0}^{\infty} \bar{\delta}_q A_q^{(\nu)}(\underline{u}'\underline{x}_j) \Big) A_p^{(\nu)}(\underline{u}'\underline{\xi})\, dU^{(\nu)}(\underline{u})$$

$$= (-1)^{(p-1)/2}\, n^{-1/2}\, \bar{\delta}_p \sum_{j=1}^{n} \int_{S_m} A_p^{(\nu)}(\underline{u}'\underline{x}_j) A_p^{(\nu)}(\underline{u}'\underline{\xi})\, dU^{(\nu)}(\underline{u})$$

$$= (-1)^{(p-1)/2} n^{-1/2} \bar{\delta}_p \sum_{j=1}^{n} A_p^{(\nu)}(\underline{\xi}'\underline{x}_j).$$

By expanding $\bar{\delta}_p$ we have

$$(3.3.1) \quad Y_p(\underline{\xi}) = \begin{cases} \pi^{-1}\, n^{-1/2}\, B(p/2, \nu+1) \sum_{j=1}^{n} A_p^{(\nu)}(\underline{\xi}'\underline{x}_j) & p = 1,3,5,\ldots, \\ 0 & p = 0,2,4,\ldots. \end{cases}$$

Under the hypothesis of uniformity we now examine the mean and covariance for $Y_p(\underline{\xi})$ in the following:

Theorem 3.3.1: Let $\underline{x}_j$ be i.i.d. $U(S_m)$, $j = 1,\ldots,n$ and $\underline{\xi},\underline{\eta} \in S_m$. Then

$$EY_p(\underline{\xi}) = 0 \qquad p = 0,1,2,\ldots,$$

$$\mathrm{Cov}(Y_p(\underline{\xi}),Y_q(\underline{\eta})) = \begin{cases} \bar{\tau}_p A_p^{(\nu)}(\underline{\xi}'\underline{\eta}) & p = q = 1,3,5,\ldots, \\ 0 & \text{otherwise,} \end{cases}$$

where $\{\bar{\tau}_p\}_{p=0}^{\infty}$ are the coefficients in the Gegenbauer transform of $\tau(w) = 2/\pi \sin^{-1}w$.

Proof: Since $Y_p(\underline{\xi}) = 0$ for p even, we need only consider p odd. By formula (2.2.61) and the computational form of $Y_p(\underline{\xi})$ given in formula (3.3.1) we get for p odd

$$EY_p(\underline{\xi}) = \pi^{-1} n^{-1/2} B(p/2,\nu+1) \sum_{j=1}^{n} EA_p^{(\nu)}(\underline{\xi}'\underline{x}_j)$$

$$= 0.$$

For the covariance, we express $Y_p(\underline{\xi})$ and $Y_p(\underline{\eta})$ in computational form by formula (3.3.1) and then apply formulas (2.2.67) and (2.2.61). Thus for p and q both odd,

$$\mathrm{Cov}(Y_p(\underline{\xi}),Y_q(\underline{\eta})) = E(Y_p(\underline{\xi})\, Y_q(\underline{\eta}))$$

$$= (-1)^{\frac{p+q}{2}} n^{-1} \bar{\delta}_p \bar{\delta}_q \sum_{i=1}^{n} \sum_{j=1}^{n} E(A_p^{(\nu)}(\underline{\xi}'\underline{x}_i)A_q^{(\nu)}(\underline{\eta}'\underline{x}_j))$$

$$= (-1)^{\frac{p+q}{2}} - 1 \; n^{-1} \; \bar{\delta}_p \bar{\delta}_q \{ \sum_{i=1}^{n} E(A_p^{(\nu)}(\underline{\xi}'\underline{x}_i) A_q^{(\nu)}(\underline{\eta}'\underline{x}_i))$$

$$+ \sum_{i \neq j} \sum E(A_p^{(\nu)}(\underline{\xi}'\underline{x}_i)) \; E(A_q^{(\nu)}(\underline{\eta}'\underline{x}_j))\}$$

$$= n^{-1} \; \bar{\delta}_p^2 \sum_{i=1}^{n} A_p^{(\nu)}(\underline{\xi}'\underline{\eta})$$

$$= \bar{\delta}_p^2 \; A_p^{(\nu)}(\underline{\xi}'\underline{\eta})$$

$$= \bar{\tau}_p \; A_p^{(\nu)}(\underline{\xi}'\underline{\eta})$$

where $\bar{\tau}_p = \bar{\delta}_p^2$ by Theorem 3.2.3.

It has already been shown that for $p = 1,3,5,\ldots$, $Y_p(\underline{\xi}) = \sum_j y_j$ where $y_j = (-1)^{(p-1)/2} \; \bar{\delta}_p \; n^{-1/2} \; A_p^{(\nu)}(\underline{\xi}'\underline{x}_j)$, $j = 1,\ldots,n$. Moreover, from the mean and variance of $Y_p(\underline{\xi})$, $p = 1,3,5,\ldots$, under uniformity we see that $Ey_j = 0$ and $Var(y_j) = \bar{\tau}_p \; n^{-1} \; A_p^{(\nu)}(1)$, $j = 1,\ldots,n$. Therefore by the central limit theorem

$$\frac{Y_p(\underline{\xi})}{(\; \bar{\tau}_p \; A_p^{(\nu)}(1) \;)^{1/2}} = \frac{\sum_{j=1}^{n} y_j}{n^{1/2} \; (\; \bar{\tau}_p \; n^{-1} \; A_p^{(\nu)}(1) \;)^{1/2}}$$

converges to $N(0,1)$ under the hypothesis of uniformity. Consequently, we have shown the following:

Theorem 3.3.2: Under the hypothesis of uniformity, the asymptotic distribution of $Y_p(\underline{\xi})$, $p = 1,3,5,\ldots$, is $N(0, \bar{\tau}_p\, A_p^{(\nu)}(1))$.

The exact distribution of $Y_p(\underline{\xi})$ is difficult to find except in the special case $m = 3$ and then only for $p = 1$. In particular, we have for $p = 1$,

$$\begin{aligned} Y_1(\underline{\xi}) &= n^{-1/2}\, \bar{\delta}_1 \sum_{j=1}^{n} A_1^{(\nu)}(\underline{\xi}'\underline{x}_j) \\ &= 2\,(\nu+1)\, n^{-1/2}\, \bar{\delta}_1 \sum_{j=1}^{n} \underline{\xi}'\underline{x}_j \\ &= 2\,(\nu+1)\, n^{-1/2}\, \bar{\delta}_1\, y \end{aligned}$$

where $y = \sum_j \underline{\xi}'\underline{x}_j$, $j = 1,\ldots,n$. To find the distribution of $Y_1(\underline{\xi})$ it is sufficient to know the distribution of y. As in section 1.5 we may define $\underline{s} = \sum_j \underline{x}_j$, $j = 1,\ldots,n$, $r_n = \|\underline{s}\|$ and $\bar{\underline{x}}_0 = \underline{s} / r_n$. Hence $y = \underline{\xi}' \sum_j \underline{x}_j = \underline{\xi}'\underline{s} = r_n\, \underline{\xi}'\bar{\underline{x}}_0$, $j = 1,\ldots,n$.

Under uniformity it is well known (see Watson (1983) for instance) that $\bar{\underline{x}}_0 \sim U(S_m)$ independently of r_n. Put $w = \underline{\xi}'\bar{\underline{x}}_0$ so that by Theorem 1.4.1, $w \sim B_{[-1,1]}(\nu+1/2, \nu+1/2)$. Thus we seek forms for the density of $y = r_n\, w$.

To have a one-to-one transformation we define $z = w$ and note that the Jacobian of the transformation from $\{r,w\}$ to $\{y,z\}$ is given by $J\{r,w \to y,z\} = z^{-1}$. We may now express $g(y)$, the density of y, as

$$g(y) = \int_{y/n}^{1} h_n(y/z)\, f(z)\, z^{-1}\, dz$$

or equivalently with $u = y/z$,

$$g(y) = \int_{y}^{n} h_n(u)\, f(y/u)\, u^{-1}\, du$$

For a random sample from a uniform distribution on S_m, $h_n(r)$, the density of the resultant length, is known to be

$$h_n(r) = r^{\nu+1}(2^\nu\Gamma(\nu+1))^{n-1}\int_0^\infty \theta^{\nu(1-n)+1}(J_\nu(\theta))^n J_\nu(\theta r)d\theta$$

for $0 < r < n$. Likewise for a random sample of observations uniformly distributed on S_m the density of $w = \underline{\xi}'\bar{\underline{x}}_0$ is

$$f(w) = \frac{(1-w^2)^{\nu-1/2}}{B(1/2,\nu+1/2)}$$

for $-1 \leq w \leq 1$.

For $m = 3$, $\underline{\xi} \in S_3$ and $\underline{x}_j$ i.i.d. $U(S_m)$, $j = 1,\ldots,n$, we have by Theorem 1.4.1 that $\underline{\xi}'\underline{x}_j$, $j = 1,\ldots,n$, are i.i.d. uniform $[-1,1]$. Thus y, the sum of n i.i.d. uniform $[-1,1]$ random variables, has density

$$g(y) = \frac{1}{2^n(n-1)!}\sum_{j=0}^{[(y+n)/2]}(-1)^j\binom{n}{j}(y+n-2j)^{n-1},$$

$-n < y < n$ and $[x]$ is the greatest integer function. Using $g(y)$ we find the density for $Y_1(\underline{\xi}) = m\, n^{-1/2}\,\bar{\underline{\delta}}_1\, y$ to be

$$f_{Y_1(\underline{\xi})}(t) = \frac{c}{2^n(n-1)!}\sum_{j=0}^{[(ct+n)/2]}(-1)^j\binom{n}{j}(ct+n-2j)^{n-1},$$

for $-n/c < t < n/c$ where $c = \{(n\pi)^{1/2}\Gamma(\nu+3/2)\} / \{2\Gamma(\nu+2)\}$.

If the modal vector $\underline{\xi} \in S_m$ is unknown, we may estimate $\underline{\xi}$ by $\bar{\underline{x}}_0$. Consequently, the distribution of $Y_p(\bar{\underline{x}}_0)$ is also of interest. Again the distribution is difficult except for $p = 1$ where

$$Y_1(\underline{\bar{x}}_0) = n^{-1/2}\,\bar{\delta}_1 \sum_{j=1}^{n} A_1^{(\nu)}(\underline{\bar{x}}_0'\underline{x}_j)$$

$$= m\,n^{-1/2}\,\bar{\delta}_1 \sum_{j=1}^{n} \underline{\bar{x}}_0'\underline{x}_j$$

$$= m\,n^{-1/2}\,\bar{\delta}_1 \left(r_n^{-1} \sum_{i=1}^{n} \underline{x}_i\right)' \sum_{j=1}^{n} \underline{x}_j$$

$$= m\,n^{-1/2}\,\bar{\delta}_1\, r_n^{-1} \sum_{i=1}^{n}\sum_{j=1}^{n} \underline{x}_i'\,\underline{x}_j$$

$$= m\,n^{-1/2}\,\bar{\delta}_1\, r_n^{-1}\, r_n^2$$

$$= m\,n^{-1/2}\,\bar{\delta}_1\, r_n.$$

Hence the density of $Y_1(\underline{\bar{x}}_0)$ may now be found from the density $h_n(r)$ to be

$$f_{Y_1(\underline{\bar{x}}_0)}(t) = c(ct)^{\nu+1}\left[2^{\nu}\,\Gamma(\nu+1)\right]^{n-1} \times \int_0^{\infty} \theta^{\nu(1-n)+1}\left[J_{\nu}(\theta)\right]^{n} J_{\nu}(\theta ct)\, d\theta$$

for $t > 0$ and where, as before, $c = \{(n\pi)^{1/2}\Gamma(\nu+3/2)\} \,/\, \{2\Gamma(\nu+2)\}$.

The asymptotic null distribution of $Y_1^2(\underline{\bar{x}}_0)$ may be found from the well known result that under uniformity, $m/n\; r_n^2 \xrightarrow{d} \chi_m^2$ as $n \to \infty$. With $m = 2(\nu + 1)$ we have $Y_1^2(\underline{\bar{x}}_0) = m^2/n\;\bar{\delta}_1^2\, r_n^2 \xrightarrow{d} m\,\bar{\delta}_1^2\,\chi_m^2$ as $n \to \infty$.

Before proceeding with examining $Y_p(\underline{\xi})$ under the alternative hypothesis of non-uniformity, we cite a relation between $Y_1^2(\underline{\xi})$ and $Y_1^2(\underline{\bar{x}}_0)$. We have

$$m \int_{S_m} Y_1^2(\underline{\xi})\, dU^{(\nu)}(\underline{\xi})$$

$$= \frac{m}{n} \bar{\delta}_1^2 \sum_{i=1}^{n} \sum_{j=1}^{n} \int_{S_m} A_1^{(\nu)}(\underline{\xi}'\underline{x}_i) A_1^{(\nu)}(\underline{\xi}'\underline{x}_j)\, dU^{(\nu)}(\underline{\xi})$$

$$= \frac{m}{n} \bar{\delta}_1^2 \sum_{i=1}^{n} \sum_{j=1}^{n} A_1^{(\nu)}(\underline{x}_i'\underline{x}_j))$$

$$= \frac{m^2}{n} \bar{\delta}_1^2 \sum_{i=1}^{n} \sum_{j=1}^{n} \underline{x}_i'\underline{x}_j$$

$$= \frac{m^2}{n} \bar{\delta}_1^2 r_n^2$$

$$= Y_1(\underline{x}_0).$$

3.4 The Statistic $Y_p(\cdot)$ under the Hypothesis of Non-Uniformity

Before turning to testing uniformity in Chapter 4 with the statistic $Y_p(\underline{\xi})$, we consider some properties of $Y_p(\underline{\xi})$ given the alternative hypothesis that the true underlying density is non-uniform. In particular, we take the alternative hypothesis, H_1, to be that the density $f(\underline{\xi}'\underline{x})$ on S_m is non-uniform, rotationally symmetric about $\underline{\xi} \in S_m$, and an increasing function of $\underline{\xi}'\underline{x}$, $\underline{x} \in S_m$. Moreover, let

$$\sum_{p=0}^{\infty} \bar{f}_p A_p^{(\nu)}(\underline{\xi}'\underline{x})$$

be the Gegenbauer transform of $f(\underline{\xi}'\underline{u})$.

Theorem 3.4.1: Suppose $\underline{x}_1,\ldots,\underline{x}_n$ are i.i.d. random vectors on S_m with common non-uniform density $f(\underline{\xi}'\underline{x})$ as described above. Let $\delta(\underline{\xi}'\underline{x})$ be the signum function and let $\bar{f}_p$ and $\bar{\delta}_p$, $p = 0,1,2,\ldots$, be the coefficients of the Gegenbauer transforms corresponding to $f(\underline{\xi}'\underline{x})$ and $\delta(\underline{\xi}'\underline{x})$. Then

$$Y_p(\underline{\xi}) = (-1)^{(p-1)/2}\, n^{-1/2}\, \bar{\delta}_p \sum_{j=0}^{n} A_p^{(\nu)}(\underline{\xi}'\underline{x}_j)$$

has mean and covariance given by

$$\mu_p = (-1)^{(p-1)/2}\, n^{1/2}\, \bar{\delta}_p\, \bar{f}_p\, A_p^{(\nu)}(1)$$

and

$$\sigma_{pq} = (-1)^{\frac{p+q}{2}} - 1\, \bar{\delta}_p\, \bar{\delta}_q \Big\{ \sum_{r=0}^{(p+q)/2} \bar{f}_{2r}\, I^{(\nu)}(p,q,2r) - \bar{f}_p\, \bar{f}_q\, A_p^{(\nu)}(1)\, A_q^{(\nu)}(1) \Big\}$$

respectively, where $p,q = 0,1,2,\ldots$ and

$$I^{(\nu)}(p,q,r) = \int_{-1}^{1} A_p^{(\nu)}(x) A_q^{(\nu)}(x) A_r^{(\nu)}(x)\, dW^{(\nu)}(x)$$

is found from formula (2.2.69).

Proof: Taking expectation with respect to the non-uniform density $f(\underline{\xi}'\underline{x}_j)$, $j=1,\ldots,n$, we find

$$EY_p(\underline{\xi}) = (-1)^{(p-1)/2}\, n^{-1/2}\, \bar{\delta}_p \sum_{j=1}^{n} EA_p^{(\nu)}(\underline{\xi}'\underline{x}_j)$$

$$= (-1)^{(p-1)/2}\, n^{-1/2}\, \bar{\delta}_p \sum_{j=1}^{n} \int_{S_m} A_p^{(\nu)}(\underline{\xi}'\underline{x}_j) \times \sum_{q=0}^{\infty} \bar{f}_q\, A_q^{(\nu)}(\underline{\xi}'\underline{x}_j)\, dU^{(\nu)}(\underline{x}_j)$$

$$= (-1)^{(p-1)/2}\, n^{-1/2}\, \bar{\delta}_p \sum_{j=1}^{n} \bar{f}_p\, A_p^{(\nu)}(1)$$

$$= (-1)^{(p-1)/2}\, n^{1/2}\, \bar{\delta}_p\, \bar{f}_p\, A_p^{(\nu)}(1).$$

To compute the covariance, σ_{pq}, we first consider $E(Y_p(\underline{\xi})\, Y_q(\underline{\xi}))$ where the expectation is again with respect to the non-uniform density $f(\underline{\xi}'\underline{x}_j)$, $j = 1,\ldots,n$. Since $Y_p(\underline{\xi}) = 0$ for $p = 0,2,4,\ldots$, then obviously $\sigma_{pq} = 0$ if either p or q is even. Thus for $p,q = 1,3,5,\ldots$,

$$E(Y_p(\underline{\xi})\, Y_q(\underline{\xi})) = E\{ (-1)^{\frac{p+q}{2} - 1}\, n^{-1}\, \bar{\delta}_p\, \bar{\delta}_q \sum_{i=1}^{n} \sum_{j=1}^{n} A_p^{(\nu)}(\underline{\xi}'\underline{x}_i) A_q^{(\nu)}(\underline{\xi}'\underline{x}_j) \}$$

$$= (-1)^{\frac{p+q}{2} - 1}\, n^{-1}\, \bar{\delta}_p\, \bar{\delta}_q\, E\{ \sum_{j=1}^{n} A_p^{(\nu)}(\underline{\xi}'\underline{x}_j) A_q^{(\nu)}(\underline{\xi}'\underline{x}_j)$$

$$+ \sum_{i \neq j} \sum A_p^{(\nu)}(\underline{\xi}'\underline{x}_i) A_q^{(\nu)}(\underline{\xi}'\underline{x}_j) \}$$

$$= (-1)^{\frac{p+q}{2} - 1}\, n^{-1}\, \bar{\delta}_p\, \bar{\delta}_q \{ \sum_{j=1}^{n} \int_{S_m} A_p^{(\nu)}(\underline{\xi}'\underline{x}_j) A_q^{(\nu)}(\underline{\xi}'\underline{x}_j)$$

$$\times \sum_{r=0}^{\infty} \bar{f}_r\, A_r^{(\nu)}(\underline{\xi}'\underline{x}_j)\, dU^{(\nu)}(\underline{x}_j)$$

$$+ \sum_{i \neq j} \sum E(A_p^{(\nu)}(\underline{\xi}'\underline{x}_i))\, E(A_q^{(\nu)}(\underline{\xi}'\underline{x}_j)) \}$$

As in the derivation of μ_p above, $E(A_p^{(\nu)}(\underline{\xi}'\underline{x}_i)) = \bar{f}_p\, A_p^{(\nu)}(1)$. Thus

$$E(Y_p(\underline{\xi})\, Y_q(\underline{\xi})) = (-1)^{\frac{p+q}{2}-1} n^{-1} \bar{\delta}_p \bar{\delta}_q \{ \sum_{j=1}^{n} \sum_{r=0}^{\infty} \bar{f}_r\, I^{(\nu)}(p,q,r)$$

$$+ \sum_{i \neq j} \bar{f}_p \bar{f}_q\, A_p^{(\nu)}(1)\, A_q^{(\nu)}(1) \}.$$

Using formula (2.2.69) and the fact that p and q are both odd, r must be even and, further, $r \leq p+q$. Hence

$$E(Y_p(\underline{\xi})\, Y_q(\underline{\xi})) = (-1)^{\frac{p+q}{2}-1} n^{-1} \bar{\delta}_p \bar{\delta}_q \{ n \sum_{r=0}^{(p+q)/2} \bar{f}_{2r}\, I^{(\nu)}(p,q,2r)$$

$$+ n(n-1) \bar{f}_p \bar{f}_q\, A_p^{(\nu)}(1)\, A_q^{(\nu)}(1) \}$$

$$= (-1)^{\frac{p+q}{2}-1} \bar{\delta}_p \bar{\delta}_q \{ \sum_{r=0}^{(p+q)/2} \bar{f}_{2r}\, I^{(\nu)}(p,q,2r)$$

$$+ (n-1) \bar{f}_p \bar{f}_q\, A_p^{(\nu)}(1)\, A_q^{(\nu)}(1) \}.$$

Then using the formula for μ_p given above,

$$\sigma_{pq} = E(Y_p(\underline{\xi})\, Y_q(\underline{\xi})) - \mu_p\, \mu_q$$

$$= (-1)^{\frac{p+q}{2}-1} \bar{\delta}_p \bar{\delta}_q \{ \sum_{r=0}^{(p+q)/2} \bar{f}_{2r}\, I^{(\nu)}(p,q,2r)$$

$$+ (n-1) \bar{f}_p \bar{f}_q\, A_p^{(\nu)}(1)\, A_q^{(\nu)}(1) - n \bar{f}_p \bar{f}_q\, A_p^{(\nu)}(1)\, A_q^{(\nu)}(1) \}$$

$$= (-1)^{\frac{p+q}{2}} - 1 \; \bar{\delta}_p \, \bar{\delta}_q \, \{ \sum_{r=0}^{(p+q)/2} \bar{f}_{2r} \, I^{(\nu)}(p,q,2r)$$

$$- \bar{f}_p \, \bar{f}_q \, A_p^{(\nu)}(1) \, A_q^{(\nu)}(1) \}.$$

For the variance of $Y_p(\underline{\xi})$ we use the notation σ_p^2 and find from Theorem 3.4.1 that

$$\sigma_p^2 = \bar{\delta}_p^2 \, \{ \sum_{r=0}^{p} \bar{f}_{2r} \, I^{(\nu)}(p,p,2r) - [\, \bar{f}_p \, A_p^{(\nu)}(1) \,]^2 \}.$$

Finally we consider the asymptotic distribution of $Y_p(\underline{\xi})$ under non-uniformity in the following:

Theorem 3.4.2: Let $f(\underline{\xi}'\underline{x})$ be a density on S_m which is non-uniform, rotationally symmetric about $\underline{\xi} \in S_m$, and an increasing function of $\underline{\xi}'\underline{x}$. Suppose $\underline{x}_1, \ldots, \underline{x}_n$ are i.i.d. random vectors on S_m with common density $f(\underline{\xi}'\underline{x})$. Then $Y_p(\underline{\xi}) = (-1)^{(p-1)/2} \, n^{-1/2} \, \bar{\delta}_p \sum_j A_p^{(\nu)}(\underline{\xi}'\underline{x}_j)$ is asymptotically distributed $N(\mu_p, \sigma_p^2)$ where μ_p and σ_p^2 are given in Theorem 3.4.1.

Proof: As in the derivation of Theorem 3.3.2, we may write $Y_p(\underline{\xi}) = \sum_j y_j$ where $y_j = (-1)^{(p-1)/2} \, \bar{\delta}_p \, n^{-1/2} \, A_p^{(\nu)}(\underline{\xi}'\underline{x}_j)$, $j = 1, \ldots, n$. Similar to the calculations in Theorem 3.4.1, we take expectations with respect to $f(\underline{\xi}'\underline{x}_j)$ to find $Ey_j = \mu_p/n$ and $Var(y_j) = \sigma_p^2/n$. Then by the central limit theorem

$$\frac{Y_p(\underline{\xi}) - \mu_p}{\sigma_p} = \frac{\sum_{j=1}^{n} (y_j - \mu_p \, n^{-1})}{n^{1/2} \, (\sigma_p \, n^{-1/2})}$$

converges in distribution to N(0,1). Thus we deduce that $Y_p(\underline{\xi}) \xrightarrow{d} N(\mu_p, \sigma_p^2)$ under non-uniformity.

CHAPTER 4
TESTING UNIFORMITY WITH THE STATISTIC $Y_p(\cdot)$

4.1 Introduction

In section 3.3. the statistic $Y_p(\underline{\xi}) = (-1)^{(p-1)/2}\, n^{-1/2}\, \bar{\delta}_p \times \sum_j A_p^{(\nu)}(\underline{\xi}'\underline{x}_j)$, $j = 1,\ldots,n$, was introduced. Section 3.3 and 3.4 found the asymptotic distribution of $Y_p(\underline{\xi})$ to be normal under uniformity and for non-uniform distributions $f(\underline{\xi}'\underline{x})$ which are rotationally symmetric about $\underline{\xi} \in S_m$ and increasing functions of $\underline{\xi}'\underline{x}$.

Here we will discuss using the statistic $Y_p(\underline{\xi})$ to test for uniformity on S_m. Indeed, we will see how Ajne's statistic, generalized to S_m, is a function of $\{Y_p^2(\underline{\xi})\}_{p>0}$.

A generalization of Ajne's statistic $I/(2\pi n)$ given by Saw (1984) was recorded in formula (1.6.5) as

$$G_n^{(\nu)} = \int_{S_m} \{ \sum_{j=1}^{n} [\phi(\underline{x}_j'\underline{\xi}) - 1]\}^2 \, dU^{(\nu)}(\underline{\xi}) \ .$$

Set $\phi(\underline{\xi}'\underline{x}_j) = 1 + \delta(\underline{\xi}'\underline{x}_j)$, $j = 1,\ldots,n$, where $\delta(w)$ is the signum function. Then

$$\begin{aligned} n^{-1}G_n^{(\nu)} &= n^{-1}\int_{S_m} \{ \sum_{j=1}^{n} \delta(\underline{x}_j'\underline{\xi})\}^2 \, dU^{(\nu)}(\underline{\xi}) \\ &= \int_{S_m} \{X(\underline{\xi})\}^2 \, dU^{(\nu)}(\underline{\xi}) \\ &= E_{\underline{\xi}}(\{X(\underline{\xi})\}^2), \qquad \underline{\xi} \in U(S_m) \end{aligned}$$

where $X(\underline{\xi}) = n^{-1/2} \sum_j \delta(\underline{\xi}'\underline{x}_j)$ is as given in formula (3.2.2). Also since $X(\underline{\xi}) = 2n^{-1/2}(N(\underline{\xi}) - n/2)$, we see that $G_n^{(v)} = 4n\, A_{m,n}$ in which $A_{m,n}$ is the generalization of Ajne's statistic to S_m given by Prentice (1978). Note that

$$\begin{aligned} X(\underline{\xi}) &= n^{-1/2} \sum_{j=1}^{n} \delta(\underline{\xi}'\underline{x}_j) \\ &= n^{-1/2} \sum_{j=1}^{n} \sum_{p=0}^{\infty} \bar{\delta}_p\, A_p^{(v)}(\underline{\xi}'\underline{x}_j) \\ &= \sum_{p=0}^{\infty} (-1)^{(p-1)/2}\, Y_p(\underline{\xi}) \,. \end{aligned}$$

Thus

$$\{X(\underline{\xi})\}^2 = \sum_{p=0}^{\infty} \{Y_p(\underline{\xi})\}^2 + \sum\sum_{p\neq q} (-1)^{\frac{p+q}{2}-1} Y_p(\underline{\xi})\, Y_q(\underline{\xi}).$$

But by formula (2.2.67), $E_{\underline{\xi}}(Y_p(\underline{\xi})\, Y_q(\underline{\xi})) = 0$ for $p \neq q$ and $\underline{\xi} \sim U(S_m)$. Therefore with $\underline{\xi} \sim U(S_m)$,

$$\begin{aligned} G_n^{(v)} &= n\, E_{\underline{\xi}}(\{X(\underline{\xi})\}^2) \\ &= n \sum_{p=0}^{\infty} E_{\underline{\xi}}(\{Y_p(\underline{\xi})\}^2) \\ &= n \sum_{p=0}^{\infty} \int_{S_m} \{Y_p(\underline{\xi})\}^2\, dU^{(v)}(\underline{u}) \end{aligned}$$

so that $G_n^{(v)}$ and $\{Y_p^2(\underline{\xi})\}_{p>0}$ are functionally related.

4.2 Optimal Properties of $Y_p(\cdot)$

Optimal properties of the statistics $Y_1(\underline{\xi})$ and $Y_1(\bar{\underline{x}}_0)$ for testing uniformity on S_m arise from the relations described in section 3.3, namely,

$$Y_1(\underline{\xi}) = m\, n^{-1/2}\, \bar{\delta}_1\, r_n\, \underline{\xi}'\bar{\underline{x}}_0$$

and

$$Y_1(\bar{\underline{x}}_0) = m\, n^{-1/2}\, \bar{\delta}_1\, r_n.$$

Thus $Y_1(\bar{\underline{x}}_0)$ enjoys the optimal properties of the Rayleigh test, while $Y_1(\underline{\xi})$ shares the optimal properties associated with $r_n\, \underline{\xi}'\bar{\underline{x}}_0 = r_n \cos\phi$ where ϕ is the angle between $\underline{\xi}$ and $\bar{\underline{x}}_0$. Although these optimal properties are widely known for the circular and spherical cases, we shall present the results for general m-spheres, $m > 2$, whenever possible.

4.2.1 Uniformity versus Langevin Distribution for Mean Direction Given

Consider the observation $\underline{u}_1, \ldots, \underline{u}_n \in S_m$ which are i.i.d. with common Langevin distribution $R^{(\nu)}(\lambda, \underline{\xi})$ where $\underline{\xi}$ is known. We propose testing H_0: $\lambda = 0$ against H_1: $\lambda > 0$ in which $\lambda = 0$ corresponds to the uniform distribution, $R^{(\nu)}(0, \underline{\xi})$, on S_m. For fixed $\lambda = \lambda_1 > 0$, the Neyman-Pearson Lemma gives the most powerful size α test as rejecting the null hypothesis for large values of the likelihood function

$$\prod_{i=1}^{n} r^{(\nu)}(\lambda_1 \underline{\xi}'\underline{u}_i) = \Lambda_\nu^{-n}(\lambda_1) \exp\Big(\lambda_1 \underline{\xi}' \sum_{i=1}^{n} \underline{u}_i\Big).$$

But this likelihood function is a monotone increasing function of $\underline{\xi}' \sum_i \underline{u}_i = r_n \underline{\xi}' \bar{\underline{x}}_0$, $i = 1,\ldots,n$. Hence the uniformly most powerful (UMP) size α test of H_0: $\lambda = 0$ against H_1: $\lambda > 0$ rejects H_0 if

$$r_n \underline{\xi}' \bar{\underline{x}}_0 > k$$

where k is chosen so that $P(r_n \underline{\xi}' \bar{\underline{x}}_0 > k \mid \lambda = 0) = \alpha$.

Equivalently, rejecting H_0 for

$$Y_1(\underline{\xi}) = m\, n^{-1/2}\, \bar{\delta}_1\, r_n \underline{\xi}' \bar{\underline{x}}_0 > c$$

where $P(Y_1(\underline{\xi}) > c \mid \lambda = 0) = \alpha$ is the UMP size α test for testing H_0: $\lambda = 0$ against H_1: $\lambda > 0$ when $\underline{\xi}$ is known.

4.2.2 Uniformity versus Langevin Distribution for Mean Direction Unknown

If $\underline{\xi}$ is unknown, we will show that rejecting the null hypothesis of uniformity, H_0: $\lambda = 0$, for r_n sufficiently large provides the uniformly most powerful invariant (UMPI) size α test of uniformity against the Langevin alternative, H_1: $\lambda > 0$.

First we fix $\lambda = \lambda_1 > 0$. Then using the Neyman-Pearson Lemma, an invariant test under orthogonal transformations is found by rejecting uniformity for large values of

$$\int_{S_m} \prod_{i=1}^{n} r^{(\nu)}(\lambda_1 \underline{\xi}' \underline{u}_i)\, dU^{(\nu)}(\underline{\xi})$$

$$= A_\nu^{-n}(\lambda_1) \int_{S_m} \exp\{\lambda_1\, \underline{\xi}' \sum_{i=1}^{n} \underline{u}_i\}\, U^{(\nu)}(\underline{u})$$

$$= A_\nu^{-n}(\lambda_1) \int_{S_m} \exp\{\lambda_1\, r_n\, \underline{\xi}' \bar{\underline{x}}_0\}\, dU^{(\nu)}(\underline{u})$$

$$= \Lambda_\nu^{-n}(\lambda_1)\ \Lambda_\nu(\lambda r_n).$$

Since for $x > 0$,

$$\frac{d}{dx}\Lambda_\nu(x) = \frac{d}{dx}\sum_{k=0}^{\infty} \frac{1}{(\nu+1)_k} \frac{(x^2/4)^k}{k!}$$

$$= \frac{2}{x}\sum_{k=1}^{\infty} \frac{1}{(\nu+1)_k} \frac{(x^2/4)^k}{(k-1)!}$$

$$= \frac{x}{2}\sum_{k=0}^{\infty} \frac{1}{(\nu+1)_{k+1}} \frac{(x^2/4)^k}{k!}$$

$$= \frac{x}{2(\nu+1)}\sum_{k=0}^{\infty} \frac{1}{(\nu+2)_k} \frac{(x^2/4)^k}{k!}$$

$$= \frac{x}{m}\ \Lambda_{\nu+1}(x)$$

$$> 0,$$

we see that $\Lambda_\nu^{-n}(\lambda_1)\ \Lambda_\nu(\lambda_1 r_n)$, $\lambda_1 > 0$, is an increasing function of r_n. Thus, rejecting uniformity when $r_n > k$ where $P(r_n > k \mid \lambda = 0) = \alpha$ provides the UMPI size α test of H_0: $\lambda = 0$ versus the Langevin alternatives H_1: $\lambda > 0$.

In section 3.3 we showed that $Y_1(\underline{\bar{x}}_0) = m\, n^{-1/2}\, \bar{\delta}_1\, r_n$. Hence rejecting H_0: $\lambda = 0$ for $Y_1(\underline{\bar{x}}_0) > c$ with $P(Y_1(\underline{\bar{x}}_0) > c \mid \lambda = 0) = \alpha$ gives the UMPI size α test against H_1: $\lambda > 0$.

4.2.3 Uniformity versus the Generalized Cardioid Distribution

Here we derive a locally most powerful (LMP) size α test prescription for testing uniformity aginst alternatives of the form

$$\phi_p(\underline{u}'\underline{\xi}) = 1 + a_p A_p^{(\nu)}(\underline{u}'\underline{\xi}) \qquad p = 1,2,3,\ldots$$

in which $\underline{\xi}$ is known. In order to insure that $\phi_p(\underline{u}'\underline{\xi})$, $p = 1,2,3,\ldots$ is a density on S_m we require $0 \leq a_p \leq 1/A_p^{(\nu)}(1)$. Since for $p = 1$, $\phi_1(\underline{u}'\underline{\xi})$ is the cardioid distribution we shall call $\phi_p(\underline{u}'\underline{\xi})$, $p = 1,2,3,\ldots$, the generalized cardioid distribution.

A test of uniformity versus the alternative $\phi_p(\underline{u}'\underline{\xi})$ may be written in the form of testing H_0: $a_p = 0$ against H_0: $a_p > 0$, $p = 1,2,3,\ldots$. To derive the LMP test, we shall need regularity conditions which permit the interchange of differentiation and integration. Then the slope of the power function may be calculated as

$$\frac{\partial}{\partial a_p} \int_C \prod_{i=1}^{n} \phi_p(\underline{\xi}'\underline{u}_i)\, dU^{(\nu)}(\underline{u}_1)\cdots U^{(\nu)}(\underline{u}_n)$$

$$= \int_C V(\underline{u}_1,\ldots,\underline{u}_n \mid a_p) \prod_{i=1}^{n} \phi_p(\underline{\xi}'\underline{u}_i)\, dU^{(\nu)}(\underline{u}_1)\cdots U^{(\nu)}(\underline{u}_n)$$

where $V(\underline{u}_1,\ldots,\underline{u}_n|a_p) = \frac{\partial}{\partial a_p} \ln \prod_{i=1}^{n} \phi_p(\underline{\xi}'\underline{u}_i)$ is the score function and C is the critical region consisting of those points $(\underline{u}_1,\ldots,\underline{u}_n)$ for which uniformity is rejected.

Verification of interchanging differentiation and integration requires $\frac{\partial}{\partial a_p} \prod_i \phi_p(\underline{\xi}'\underline{u}_i)$ to be bounded for $0 \leq a_p \leq 1/A_p^{(\nu)}(1)$, $p = 1,2,3,\ldots$. Hence, for $p = 1,2,3,\ldots$,

$$\left| \frac{\partial}{\partial a_p} \prod_{i=1}^{n} \phi_p(\xi' u_i) \right| = \left| \frac{\partial}{\partial a_p} \prod_{i=1}^{n} \{1 + a_p A_p^{(\nu)}(\xi' u_i)\} \right|$$

$$= \left| \sum_{i=1}^{n} A_p^{(\nu)}(\xi' u_i) \prod_{\substack{j=1 \\ j \neq i}}^{n} \{1 + a_p A_p^{(\nu)}(\xi' u_i)\} \right|$$

$$\leq \sum_{i=1}^{n} A_p^{(\nu)}(1) \prod_{\substack{j=1 \\ j \neq i}}^{n} \{1 + a_p A_p^{(\nu)}(1)\}$$

$$\leq n A_p^{(\nu)}(1) 2^{n-1}.$$

Now let f_0 and f_1 be real-valued integrable functions relative to a measure μ. Then an extension of the Neyman-Pearson Lemma (see Fraser (1976)) gives $f_1(\underline{y}) > k f_0(\underline{y})$ as the critical region, C, which maximizes $\int_C f_1(\underline{y}) \, d\mu(\underline{y})$ subject to $\int_C f_0(\underline{y}) \, d\mu(\underline{y}) = \alpha$.

Applying this extended version of the Neyman-Pearson Lemma to the slope of the power function, we find the LMP size α test will reject uniformity for large values of the score function evaluated at $a_p = 0$. That is, the LMP test of size α has critical region

$$C = \{(\underline{u}_1, \ldots, \underline{u}_n) : V(\underline{u}_1, \ldots, \underline{u}_n | a_p) \Big|_{a_p=0} > k\}$$

where

$$P(V(\underline{u}_1, \ldots, \underline{u}_n | a_p) \Big|_{a_p=0} > k) = \alpha, \quad p = 1,2,3,\ldots.$$

The score function evaluated at $a_p = 0$, $p = 1,2,3,\ldots$, is

$$V(\underline{u}_1, \ldots, \underline{u}_n | a_p) \Big|_{a_p=0} = \frac{\partial}{\partial a_p} \ln \prod_{i=1}^{n} \phi_p(\xi' u_i) \Big|_{a_p=0}$$

$$= \sum_{i=1}^{n} \frac{A_p^{(\nu)}(\underline{\xi}'\underline{u}_i)}{1 + a_p A_p^{(\nu)}(\underline{\xi}'\underline{u}_i)} \Big|_{a_p=0}$$

$$= \sum_{i=1}^{n} A_p^{(\nu)}(\underline{\xi}'\underline{u}_i)$$

For $p = 1,2,3,\ldots$, set $T_p = n^{-1/2} \sum_i A_p^{(\nu)}(\underline{\xi}'\underline{u}_i)$. We may use T_p directly as the test statistic and reject uniformity for large values of T_p. Similar to the results for the mean, variance and asymptotic null distribution of $Y_p(\underline{\xi})$, we find that under uniformity $T_p \sim N(0, A_p^{(\nu)}(1))$ as $n \to \infty$. For $f(\underline{\xi}'\underline{u}_i)$ a continuous non-uniform, rotationally symmetric, unimodal alternative, $T_p \sim N(\mu, \sigma^2)$ as $n \to \infty$ where

$$\mu = n^{1/2} \bar{f}_p A_p^{(\nu)}(1) \quad \text{and} \quad \sigma^2 = \sum_{r=0}^{p} \bar{f}_{2r} I^{(\nu)}(p,p,2r) - (\bar{f}_p A_p^{(\nu)}(1))^2.$$

Since for $p = 1,3,5,\ldots$,

$$\sum_{i=1}^{n} A_p^{(\nu)}(\underline{\xi}'\underline{u}_i) = \pi\, n^{-1/2}\, \delta_p^{-1}\, Y_p(\underline{\xi}).$$

we may use the results already derived for $Y_p(\underline{\xi})$. Thus a LMP size α test rejects H_0: $a_p = 0$, $p = 1,3,5,\ldots$, for $Y_p(\underline{\xi}) > k$ with $P(Y_p(\underline{\xi}) > k \mid a_p = 0) = \alpha$.

As a special case, we observe that $\phi_1(\underline{\xi}'\underline{u}) = 1 + a_1 A_1^{(\nu)}(\underline{\xi}'\underline{u})$, $\underline{\xi},\underline{u} \in S_m$, $0 < a_1 < 1/m$ is the cardioid distribution on S_m. Consequently for $\underline{\xi}$ given, rejecting H_0: $a_1 = 0$ whenever $Y_1(\underline{\xi}) > k$ with $P(Y_1(\underline{\xi}) > k \mid a_1 = 0) = \alpha$ provides the LMP size α test against the cardioid alternatives H_1: $0 < a_1 < 1/m$.

4.2.4 Uniformity versus other Selected Distributions

With only slight modifications, the derivation in subsection 4.2.3 may be adapted to derive a LMP test of uniformity against the alternative

$$\phi_{s,b}(\underline{\xi}'\underline{u}) = 1 + b\left\{\sum_{p=0}^{s} a_p A_p^{(\nu)}(\underline{\xi}'\underline{u})\right\}$$

for $\underline{\xi}$ given and $0 < b < 1$. In order for $\phi_{s,b}(\underline{\xi}'\underline{u})$ to be a density on S_m, the coefficients a_p, $p = 0,\ldots,s$, must be sufficiently small.

The testing problem may be formulated as testing H_0: $b = 0$ against H_1: $0 < b < 1$. The score function $V(\underline{u}_1,\ldots,\underline{u}_n \mid b)$ evaluated at $b = 0$ becomes

$$V(\underline{u}_1,\ldots,\underline{u}_n \mid b)\Big|_{b=0} = \sum_{p=0}^{s} a_p \sum_{i=1}^{n} A_p^{(\nu)}(\underline{\xi}'\underline{u}_i) .$$

Then with $T_p = n^{-1/2}\sum_i A_p^{(\nu)}(\underline{\xi}'\underline{u}_i)$, a LMP size α test may reject uniformity for $\sum_p a_p n^{1/2} T_p > k$, $p = 0,\ldots,s$, in which $P(\sum_p a_p n^{1/2} T_p > k \mid b = 0) = \alpha$.

If the even numbered coefficients of $\phi_{s,b}(\underline{\xi}'\underline{u})$ are zero, define

$$\phi^*_{s,b}(\underline{\xi}'\underline{u}) = 1 + b\{\sum_{p=0}^{s} a_{2p+1} A_{2p+1}^{(\nu)}(\underline{\xi}'\underline{u})\}.$$

Hence,

$$V(\underline{u}_1,\ldots,\underline{u}_n \mid b)\Big|_{b=0} = \sum_{p=0}^{s} c_{2p+1} Y_{2p+1}(\underline{\xi})$$

where $c_{2p+1} = a_{2p+1} \pi n^{1/2} \bar{\delta}^{-1}_{2p+1}$, $p = 0,1,\ldots,s$. Then the critical region for a LMP size α test of H_0: $b = 0$ against H_1: $0 < b < 1$ rejects for

$$\sum_{p=0}^{s} c_{2p+1} Y_{2p+1}(\underline{\xi}) > k$$

such that

$$P\left(\sum_{p=0}^{s} c_{2p+1} Y_{2p+1}(\underline{\xi}) > k \mid b = 0\right) = \alpha.$$

In addition, $Y_1(\underline{\bar{x}}_0)$ shares one further optimal property with the Rayleigh test. Using the conventional polar angle notation as in section 1.4, we may project the bivariate normal distribution $N_2(\underline{n},\Sigma)$ onto the circle S_2 to obtain the density of the offset normal distribution on S_2 as

$$g_\Delta(\theta) = \exp(-1/2\ \Delta^2) + \Delta\sin\theta\ \phi(\Delta\cos\theta)\ \Phi(\Delta\sin\theta), \quad (dU^{(0)})$$

$0 \leq \theta < 2\pi$, where $\phi(\cdot)$ and $\Phi(\cdot)$ are the standard normal density and distribution functions respectively and $\Delta^2 = \underline{n}'\Sigma^{-1}\underline{n}$. Bhattacharyya and Johnson (1969) have shown that the Rayleigh test and hence $Y_1(\underline{\bar{x}}_0)$ is locally most powerful invariant for testing uniformity, H_0: $\Delta = 0$, versus the offset normal alternatives, H_1: $\Delta \neq 0$.

4.3 Power Computations

We have noted optimal properties of the test statistics $Y_p(\underline{\xi})$ and $Y_p(\bar{\underline{x}}_0)$ for the Langevin, generalized cardioid, offset normal and other selected distributions. For the alternatives concerning Saw's distribution, Saw's hyperspherical t distribution, the Cauchy distribution and the wrapped normal distribution, tests involving $Y_p(\underline{\xi})$ or $Y_p(\bar{\underline{x}}_0)$ for uniformity with optimal properties are presently not known.

However, for $\underline{\xi}$ known the statistic $Y_1(\underline{\xi})$ appears to have good power for the group of alternatives listed above. For these alternatives, there seems to be little, if any, gain in power by considering either a linear combination of $Y_1(\underline{\xi})$ and $Y_3(\underline{\xi})$ or a linear combination of their squares.

For large samples and $\underline{\xi}$ known, we may reject uniformity when $Y_p(\underline{\xi}) > c$ where $c = z_\alpha(\bar{\tau}_1 A_1^{(\nu)}(1))^{1/2}$ and z_α is the upper 100α percentage point of the standard normal distribution. For large samples the power is then given by

$$P_{H_1}(Y_1(\underline{\xi}) > c) \cong 1 - \Phi((c-\mu_1)/\sigma_1)$$

where $\Phi(\cdot)$ is the standard normal distributin function while μ_1 and σ_1^2 are defined in Theorem 3.4.1. In the following tables we present powers of tests rejecting uniformity for $Y_1(\underline{\xi}) > c$ against the alternatives mentioned above. Rejecting for $Y_1(\underline{\xi}) > c$ seems to perform well for the Cauchy and wrapped normal distribution but less well for Saw's distribution.

Table 4.1 Powers of the Test $Y_1(\underline{\xi}) > c$, $\alpha = .05$, for Testing Uniformity against the Cauchy Distribution with Parameter λ

n = 15

λ	m	2	3	4	5	10	15
0.1		.135	.164	.191	.217	.336	.442
0.2		.287	.378	.461	.535	.794	.917
0.3		.499	.650	.762	.841	.983	.999
0.4		.724	.871	.944	.976	1.000	1.000
0.5		.897	.976	.995	.999	1.000	1.000
0.6		.980	.999	1.000	1.000	1.000	1.000
0.7		.999	1.000	1.000	1.000	1.000	1.000

n = 25

λ	m	2	3	4	5	10	15
0.1		.173	.217	.258	.298	.474	.615
0.2		.407	.535	.641	.727	.939	.989
0.3		.691	.841	.922	.963	.999	1.000
0.4		.902	.976	.995	.999	1.000	1.000
0.5		.985	.999	1.000	1.000	1.000	1.000
0.6		.999	1.000	1.000	1.000	1.000	1.000

Table 4.2 Powers of the Test $Y_1(\underline{\xi}) > c$, $\alpha = .05$, for Testing Uniformity against Saw's Distribution with Parameter λ

$n = 15$

λ	m	2	3	4	5	10	15
0.1		.122	.126	.128	.130	.133	.134
0.2		.242	.256	.264	.269	.279	.283
0.3		.408	.435	.449	.458	.478	.485
0.4		.596	.631	.648	.661	.684	.692
0.5		.762	.802	.818	.830	.846	.852
0.6		.893	.917	.927	.933	.942	.945
0.7		.963	.974	.979	.981	.984	.985
0.8		.991	.995	.996	.996	.997	.997
0.9		.999	.999	.999	1.000	1.000	1.000

$n = 25$

λ	m	2	3	4	5	10	15
0.1		.153	.159	.162	.164	.169	.171
0.2		.340	.360	.371	.378	.393	.398
0.3		.580	.613	.630	.640	.661	.669
0.4		.798	.828	.842	.850	.867	.872
0.5		.932	.948	.955	.959	.966	.968
0.6		.985	.991	.992	.993	.995	.995
0.7		.998	.999	.999	.999	1.000	1.000

Table 4.3 Powers of the Test $Y_1(\underset{\sim}{x}) > c$, $\alpha = .05$, for Testing Uniformity against Saw's Hyperspherical t-distribution with Parameter λ

n = 15

λ	m	2	3	4	5	10	15
0.1		.277	.255	.243	.236	.217	.209
0.2		.374	.337	.314	.298	.260	.244
0.3		.470	.421	.388	.365	.308	.283
0.4		.553	.500	.461	.433	.357	.323
0.5		.618	.569	.528	.496	.407	.364
0.6		.664	.624	.585	.553	.454	.405
0.7		.697	.665	.631	.600	.498	.444
0.8		.714	.692	.663	.635	.536	.479
0.9		.724	.708	.683	.658	.564	.508

n = 25

λ	m	2	3	4	5	10	15
0.1		.306	.279	.263	.253	.229	.219
0.2		.439	.389	.358	.338	.287	.265
0.3		.565	.501	.458	.428	.350	.316
0.4		.667	.603	.554	.517	.415	.369
0.5		.742	.686	.637	.597	.480	.423
0.6		.791	.748	.704	.666	.540	.475
0.7		.822	.791	.754	.719	.594	.524
0.8		.840	.818	.788	.757	.638	.567
0.9		.848	.833	.807	.781	.672	.601

Table 4.4 Powers of the Test $Y_1(\underline{\xi}) > c$, $\alpha = .05$, for Testing Uniformity against the Wrapped Normal Distribution with Parameter λ

n = 15

λ	m	2	3	4	5	10	15
0.1		.134	.161	.187	.212	.329	.438
0.2		.283	.373	.458	.539	.848	.981
0.3		.499	.665	.798	.894	1.000	1.000
0.4		.742	.915	.986	1.000	1.000	1.000
0.5		.928	.997	1.000	1.000	1.000	1.000
0.6		.995	1.000	1.000	1.000	1.000	1.000

n = 25

λ	m	2	3	4	5	10	15
0.1		.172	.215	.255	.294	.473	.624
0.2		.405	.537	.650	.745	.974	1.000
0.3		.670	.865	.952	.987	1.000	1.000
0.4		.921	.992	1.000	1.000	1.000	1.000
0.5		.994	1.000	1.000	1.000	1.000	1.000

CHAPTER 5
GRAPHICAL DENSITY ESTIMATION

5.1 Introduction

To this point we have been concerned with testing uniformity on S_m against certain rotationally symmetric, unimodal alternatives. Besides conducting tests of uniformity, in practice it would be desirable to be able to suggest from which particular distribution we are sampling.

Here we present a graphical curve fitting aid for distributions on S_m, similar to that for the Pearson frequency curves, i.e. Types I-VII, as given by Pearson and Hartley (1966). As an aid to determining the particular Pearson curve, a graph of $\beta = \mu_3^2 / \mu_2^3$ plotted against $\beta = \mu_4 / \mu_2^2$, where μ_r is the r^{th} moment about the mean. In our case, a distribution on S_m may be determined from a graph of $\bar{f}_1^{(\nu)}$ plotted against $\bar{f}_2^{(\nu)}$ where $\nu = m/2 - 1$ and $\bar{f}_p^{(\nu)}$ is the p^{th} coefficient in the Gegenbauer transform of the density $f(\lambda \underline{\xi}'\underline{u})$. Since most practical applications dictate data in two or three dimensions, we report curve fitting graphs for $m = 2$ and $m = 3$ in Figures 5.1 and 5.2 respectively.

5.2 A Graphical Curve Fitting Aid

Suppose that $f(\lambda \underline{\xi}'\underline{u})$ is a density on S_m with Gegenbauer transform

$$\sum_{p=1}^{\infty} \bar{f}_p^{(\nu)} A_p^{(\nu)}(\lambda \underline{\xi}'\underline{u}).$$

The coefficients of the Gegenbauer transforms for many distributions on S_m are given in section 2.3. At the end of this section Figures 5.1 and 5.2 give graphs of $\bar{f}_1^{(\nu)}$ plotted against $\bar{f}_2^{(\nu)}$ for m = 2 and m = 3 respectively. Given the random sample $\underline{u}_1,\ldots,\underline{u}_n$ from S_m we may estimate $\bar{f}_p^{(\nu)}$ by

$$\hat{f}_p^{(n)} = \left(n\, A_p^{(\nu)}(1)\right)^{-1} \sum_{j=1}^{n} A_p^{(\nu)}(\underline{\xi}'\underline{u}_j).$$

With $\nu = m/2 - 1$ determined by context, we henceforth discontinue the superscript (ν) notation for $\bar{f}_p^{(\nu)}$ and $\hat{f}_p^{(\nu)}$.

The quantity $\hat{f}_p$ is unbiased for $\bar{f}_p$, p > 1, as is seen from

$$\begin{aligned} E\hat{f}_p &= \left(n\, A_p^{(\nu)}(1)\right)^{-1} \sum_{j=1}^{n} EA_p^{(\nu)}(\underline{\xi}'\underline{u}_j) \\ &= \left(nA_p^{(\nu)}(1)\right)^{-1} \sum_{j=1}^{n} \int_{S_m} A_p^{(\nu)}(\underline{\xi}'\underline{u}_j) \sum_{r=0}^{\infty} \bar{f}_r\, A_r^{(\nu)}(\underline{\xi}'\underline{u}_j)\, dU^{(\nu)}(\underline{u}_j) \\ &= \left(nA_p^{(\nu)}(1)\right)^{-1} \sum_{j=1}^{n} \bar{f}_p \int_{S_m} \left[A_p^{(\nu)}(\underline{\xi}'\underline{u}_j)\right]^2 dU^{(\nu)}(\underline{u}_j) \\ &= \bar{f}_p. \end{aligned}$$

The second moment of $\hat{f}_p$, p > 1, is found to be

$$\begin{aligned} E\hat{f}_p^2 &= \left(n\, A_p^{(\nu)}(1)\right)^{-2} E\left\{\sum_{j=1}^{n} A_p^{(\nu)}(\underline{\xi}'\underline{u}_j)\right\}^2 \\ &= \left(n\, A_p^{(\nu)}(1)\right)^{-2} \Big\{\sum_{j=1}^{n} E\left(A_p^{(\nu)}(\underline{\xi}'\underline{u}_j)\right)^2 \\ &\qquad + \sum_{i\neq j}\sum EA_p^{(\nu)}(\underline{\xi}'\underline{u}_i)\, EA_p^{(\nu)}(\underline{\xi}'\underline{u}_j)\Big\} \end{aligned}$$

$$= (n A_p^{(\nu)}(l))^{-2} \{ \sum_{j=1}^{n} \int_{S_m} [A_p^{(\nu)}(\underline{\xi}'\underline{u}_j)]^2$$

$$\times \sum_{r=0}^{\infty} \bar{f}_r A_p^{(\nu)}(\underline{\xi}'\underline{u}_j)\, dU^{(\nu)}(\underline{u}_j)$$

$$+ \sum_{i \neq j} \sum [\int_{S_m} A_p^{(\nu)}(\underline{\xi}'\underline{u}_i) \sum_{r=0}^{\infty} [\int_{S_m} \bar{f}_r A_p^{(\nu)}(\underline{\xi}'\underline{u}_i)\, dU^{(\nu)}(\underline{u}_i)]$$

$$\times [\int_{S_m} A_p^{(\nu)}(\underline{\xi}'\underline{u}_j) \sum_{r=0}^{\infty} \bar{f}_r A_r^{(\nu)}(\underline{\xi}'\underline{u}_j)]\}$$

$$= (n A_p^{(\nu)}(l))^{-2} \{ \sum_{j=1}^{n} \sum_{r=0}^{p} \bar{f}_{2r} \int_{S_m} [A_p^{(\nu)}(\underline{\xi}'\underline{u}_j)]^2$$

$$\times A_{2r}^{(\nu)}(\underline{\xi}'\underline{u}_j)\, dU^{(\nu)}(\underline{u}_j)$$

$$+ \sum_{i \neq j} \sum [\bar{f}_p \int_{S_m} A_p^{(\nu)}(\underline{\xi}'\underline{u}_i)\, A_p^{(\nu)}(\underline{\xi}'\underline{u}_i)\, dU^{(\nu)}(\underline{u}_i)]$$

$$\times [\bar{f}_p \int_{S_m} A_p^{(\nu)}(\underline{\xi}'\underline{u}_j)\, A_p^{(\nu)}(\underline{\xi}'\underline{u}_j)\, dU^{(\nu)}(\underline{u}_j)]\}$$

$$= (n A_p^{(\nu)}(l))^{-2} \{ \sum_{j=1}^{n} \sum_{r=0}^{p} \bar{f}_{2r} I^{(\nu)}(p,p,2r)$$

$$+ \sum_{i \neq j} \sum [\bar{f}_p A_p^{(\nu)}(l)]^2\}$$

$$= n^{-1}\{[A_p^{(\nu)}(l)]^{-2} \sum_{r=0}^{p} \bar{f}_{2r} I^{(\nu)}(p,p,2r) + (n-1)\, \bar{f}_p^2\}$$

where $I^{(\nu)}(p,q,r) = \int_{S_m} A_p^{(\nu)}(\underline{\xi}'\underline{u})\, A_q^{(\nu)}(\underline{\xi}'\underline{u})\, A_r^{(\nu)}(\underline{\xi}'\underline{u})\, dU^{(\nu)}(\underline{u})$ is given by formula 2.2.69. Denoting the variance of $\hat{f}_p$ by γ^2 we have,

$$\gamma^2 = n^{-1}\{[A_p^{(\nu)}(l)]^{-2} \sum_{r=0}^{p} \bar{f}_{2r} I^{(\nu)}(p,p,2r) - \bar{f}_p^2\}.$$

Then by the central limit theorem we find,

$$\frac{\hat{f}_p - \bar{f}_p}{\gamma} \xrightarrow{d} N(0,1) \qquad \text{as } n \to \infty.$$

Suppose the random sample $\underline{u}_1, \ldots, \underline{u}_n$ is given. In practice, the modal vector, $\underline{\xi}$, is usually unknown so that we may estimate $\underline{\xi}$ by $\bar{\underline{x}}_0$, the sample mean direction, as defined in section 1.5. We may then use

$$\hat{f}_p^* = \left(n\, A_p^{(\nu)}(1)\right)^{-1} \sum_{j=1}^{n} A_p^{(\nu)}(\bar{\underline{x}}_0' \underline{x}_j)$$

to estimate $\bar{f}_p$.

For $f(\lambda \underline{\xi}' \underline{u})$ a density on S_m we may construct a graph of $\bar{f}_1$ versus $\bar{f}_2$ for various values of λ. In section 2.3 expressions are given for $\bar{f}_1$ and $\bar{f}_2$ for the Langevin, Saw's, Saw's hyperspherical t, Cauchy, wrapped normal, cardioid and point distributions. Figures 5.1 and 5.2 present such graphs of $\bar{f}_1$ versus $\bar{f}_2$ for m = 2 and m = 3 respectively. By computing $\hat{f}_1^*$ and $\hat{f}_2^*$ and observing which graphed curve is closest to the point $(\hat{f}_1^*, \hat{f}_2^*)$, one may decide upon a distribution to use to fit the data.

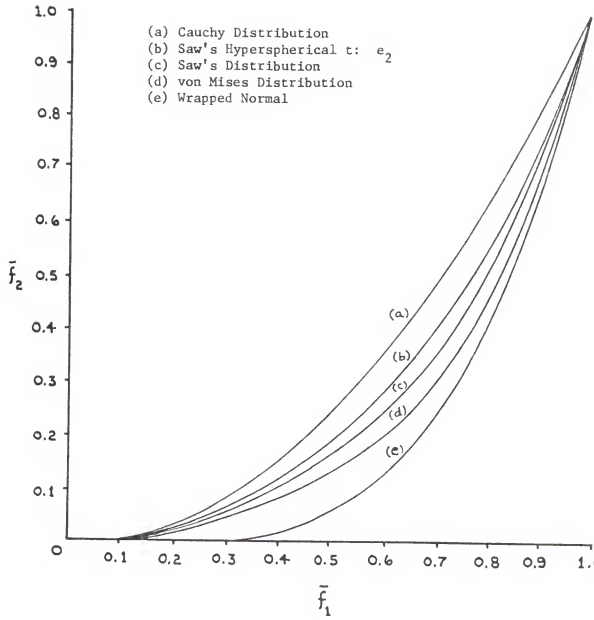

Figure 5.1 Graph of $\bar{f}_1$ versus $\bar{f}_2$ for

Selected Distributions with m=2

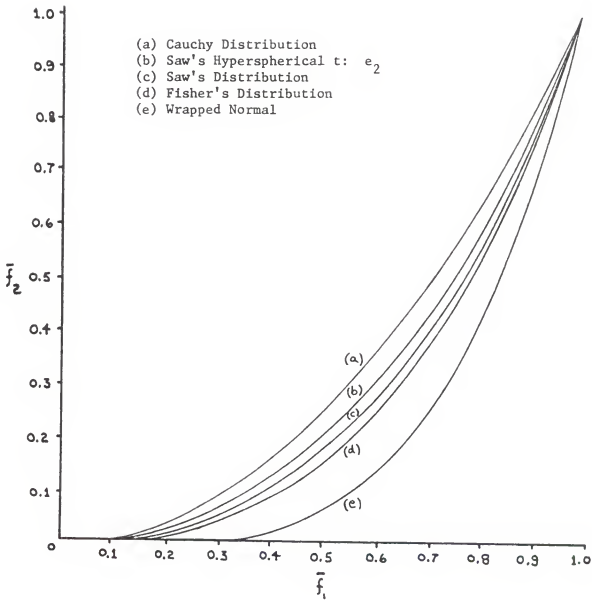

Figure 5.2 Graph of $\bar{f}_1$ versus $\bar{f}_2$ for

Selected Distributions with m=3

CHAPTER 6
SUMMARY

A brief introduction to directional data was presented in Chapter 1, section 1.1 and there it was argued that the main importance of dealing with methods applicable to general m-spheres is to unify present methods now treated separately on the circle and sphere. Some areas where directional data methods find application were indicated in section 1.2.

In section 1.3 we defined uniform measure on the m-sphere, S_m, and densities relative to this uniform measure. Several useful distributional theorems were presented in section 1.4 along with a catalog of common distributions, namely, the Langevin distribution, Saw's distribution, Saw's hyperspherical t-distribution, the circular cardioid distribution, the circular point distribution and the circular wrapped normal distribution. Where appropriate, certain properties of these distributions were recorded.

Section 1.5 pointed out that the uniform distribution in directional data occupies a more prominent position than does its uniform counterpart in the linear case. Indeed, under mild conditions, the resultant length, r_n, and the mean direction, $\bar{x}_0$, are independent if if only if we are sampling from the uniform density on S_m. In addition, the uniform distribution appears as the limiting distribution in the central limit theorem on the circle.

Several well known tests applicable to S_m for testing uniformity were stated in section 1.6. These tests include Rayleigh's test, a generalized version of Agne's test and a generalized version of a class of tests given by Beran (1968, 1969).

Having surveyed the current tests for uniformity in section 1.6, we presented the foundation for density representation on S_m in section 2.1. It was shown that Gegenbauer polynomials, defined in section 2.2, provided a means to represent densities $f(\underline{x}'\underline{\xi})$ which have rotational symmetry and which are unimodal.

The definition of Gegenbauer polynomials given in section 2.2 used a standardization and weight function proposed by Saw (1984). Here it was seen that this particular standardization and weight function simplified many common results including the integrated square of a Gegenbauer polynomial. The remainder of section 2.2 was devoted to listing many standard results for Gegenbauer polynomials in terms of this standardization and weight function.

Section 2.3 defined the Gegenbauer transform of a function and listed in Table 2.5 the coefficients of the Gegenbauer transform of some common densities.

Due to the complicated asymptotic distribution of Ajne's test statistic generalized to m-dimensions, Chapter 3 proposed a new test statistic for testing uniformity on S_m. This new test statistic was shown to have a normal asymptotic distribution. But before giving this new test statistic, section 3.2 defined $X(\underline{u})$ to be a standardized version of $N(\underline{u})$, the number of sample points in some m-dimensional hemisphere, and defined $\delta(\underline{u}'\underline{\xi})$ to be the signum function. The mean and covariance were computed separately for the functions $X(\underline{u})$ and $\delta(\underline{u}'\underline{\xi})$.

As a final preliminary result, the coefficients for the Gegenbauer transform of $\tau(\underline{u}'\underline{w})$, the covariance between $X(\underline{u})$ and $X(\underline{w})$, was given.

Section 3.3 defined the new test statistic $Y_p(\underline{\xi})$ and gave a computational form for $Y_p(\underline{\xi})$ as well. Under the null hypothesis of uniformity, the mean and covariance for $Y_p(\underline{\xi})$ were derived along with the fact that $Y_p(\underline{\xi})$ has an asymptotic normal distribution. The exact distribution of $Y_1(\underline{\xi})$ was presented for $m = 3$. The exact distribution of $Y_1(\underline{\bar{x}}_0)$ was also given under the hypothesis of uniformity.

For non-uniform rotationally symmetric unimodal alternatives, section 3.4 listed the mean and covariance of $Y_p(\underline{\xi})$ and again found that $Y_p(\underline{\xi})$ has a normal asymptotic distribution.

Chapter 4 discussed testing uniformity with $Y_p(\underline{\xi})$. Section 4.1 showed a relationship between the statistics $\{Y_p^2(\underline{\xi})\}$ and the generalization of Ajne's statistic as given by Saw (1984).

Optimal properties of certain tests involving the statistics $Y_p(\underline{\xi})$ or $Y_p(\underline{\bar{x}}_0)$ were shown in section 4.2. There we derived UMP and UMPI tests of uniformity versus the Langevin distributin for mean direction known and unknown respectively. We gave LMP tests against the generalized cardioid alternative and other selected alternatives.

For some alternatives, tests for uniformity involving $Y_p(\underline{\xi})$ or $Y_p(\underline{\bar{x}}_0)$ which have optimal properties were not known. In these cases section 4.3 examined the power of $Y_1(\underline{\xi})$ which seemed to perform well or moderately well against these alternatives.

In the course of these investigations, it was found desirable to be able to suggest from wich particular distribution a sample was taken. Thus a graphical curve fitting aid was presented in Chapter 5.

Of future interest would be to more closely examine the power of certain tests involving $Y_p(\underline{\xi})$ or $Y_p(\underline{\bar{x}}_0)$. For instance, if we reject for large values of $c_1Y_1(\underline{\xi}) + c_3Y_3(\underline{\xi})$, then to choose c_1 and c_3 to maximize the power involves minimizing the function

$$\frac{{}_0\sigma_{13}}{\sigma_{13}}\left\{z_\alpha - \frac{c_1\mu_1 + c_3\mu_3}{{}_0\sigma_{13}}\right\}$$

where μ_1 and μ_3 are the respective means of $Y_1(\underline{\xi})$ and $Y_3(\underline{\xi})$ while ${}_0\sigma_{13}$ and σ_{13} represent the covariance between $Y_1(\underline{\xi})$ and $Y_3(\underline{\xi})$ under the null and alternative hypotheses respectively.

It would also be of interest to perform some simulation studies to compare this new test statistic with the generalized version of Ajne's statistic and to check the rate of convergence to normality of the distribution of $Y_p(\underline{\xi})$.

APPENDIX: INDEX OF NOTATIONS

Definiendum	Definiens	Page Introduced
E^m	Euclidean space of m dimensions	1
$\underline{x}'\underline{y}$	inner product: $\underline{x}'\underline{y} = \sum_i x_i y_i$	1
S_m	surface of the unit m-sphere: $\{\underline{x}: \underline{x} \in E^m, \underline{x}'\underline{x} = 1\}$	1
$\underline{x}$	column vector	1
$C_m(\underline{\xi},u)$	surface "cap": $\{\underline{x}: \underline{x} \in S_m, \underline{x}'\underline{\xi} > u\}$	4
A_m	collection of all surface "caps": $\{C_m(\underline{\xi},u): \underline{\xi} \in S_m, \lvert u\rvert < 1\}$	4
$\lvert u\rvert$	absolute value of u	5
B_m	σ-algebra generated by A_m	5
$U^{(\nu)}$	uniform measure on S_m	5
ν	$\nu = m/2 - 1$	5
B(a,b)	beta function: $B(a,b) = \int_{-1}^{1} x^{a-1}(1-x)^{b-1}\,dx$	5
P(A)	probability of the event A	5
$W^{(\nu)}(u)$	a measure on $[-1,1]$	5
a.e.	almost everywhere with respect to a measure μ: except on a set of μ-measure zero	7
$\sim$	as in $\underline{u} \sim U(S_m)$: $\underline{u}$ is distributed uniformly on S_m (symbol may be used with other distributions also)	7
$U(S_m)$	uniform distribution on S_m	7
i_m	invariant measure on S_m: $i_m(\cdot) = \alpha_m U^{(\nu)}(\cdot)$	7
α_m	surface area of the unit m-sphere: $\alpha_m = 2\pi^{m/2}/\Gamma(m/2)$	7
$\underline{0}$	vector which has all its elements equal to zero	7
I_m	identity matrix of dimension $m \times m$	7
$N_m(\underline{\mu},V)$	m-variate normal distributin with mean vector $\underline{\mu}$ and covariance matrix V	7

Definiendum	Definiens	Page Introduced
$\|\underline{x}\|$	Euclidean norm: $\|\underline{x}\| = (\underline{x}'\underline{x})^{1/2}$	7
χ^2_m	chi-square distribution with m degrees of freedom	7
$J\{\ \to\ \}$	as in $J\{\underline{z} \to \underline{r},\underline{u}\}$: Jacobian of transformation from $\underline{z}$ to $\{\underline{r},\underline{u}\}$	7
$(dr\ di_m)$	indication of differential elements	8
$\Gamma(t)$	gamma function: $\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}\,dx$	8
$[a,b]$	interval from a to b	8
$B_{[a,b]}(p,q)$	beta distribution on the interval [a,b] with parameters p and q	8
$(P)_{\cdot i}$	i^{th} column of the matrix P	9
x_i	i^{th} element of the vector $\underline{x}$	9
$E_{\underline{u}}(\cdot)$	expectation with respect to the distribution of $\underline{u}$	10
t_m	Student's t-distribution with m degrees of freedom	11
i.i.d.	independent identically distributed	11
$Cov(\underline{x})$	covariance matrix of the random vector $\underline{x}$	11
$\overset{d}{\to}$	convergence in distribution	11
$r^{(\nu)}(\lambda\underline{\xi}'\underline{u})$	Langevin density on S_m	12
$\Lambda_\nu(g)$	$\Lambda_\nu(\lambda) = (2/\lambda)^\nu\,\Gamma(\nu+1)\,I_\nu(\lambda)$	12
$I_\nu(\lambda)$	modified Bessel function of order ν	12
${}_pF_q(\underline{a};\underline{b};z)$	generalized hypergeometric function where $${}_pF_q(\underline{a};\underline{b};z) = \sum_{n=0}^{\infty} \frac{(a_1,n)\cdots(a_p,n)}{(b_1,n)\cdots(b_q,n)} \frac{z^n}{n!}$$	13
$r^{(0)}(\lambda\underline{\xi}'\underline{u})$	von Mises density on the circle	13
$r^{(1/2)}(\lambda\underline{\xi}\underline{u})$	Fisher density on the 3-sphere	13
$R^{(\nu)}(\lambda,\underline{\xi})$	Langevin distribution on S_m	13

Definiendum	Definiens	Page Introduced
$\rightarrow$	limit symbol as in $\lambda \rightarrow 0$ means the limit as λ tends to zero	13
$j^{(\nu)}(\lambda\underline{\xi}'\underline{u})$	density for Saw's distribution on S_m	14
$J^{(\nu)}(\lambda\underline{\xi}'\underline{u})$	Saw's distribution on S_m	14
$e_s^{(\nu)}(\lambda\underline{\xi}'\underline{u})$	density for Saw's hyperspherical t-distribution	15
$K_s(\lambda)$	$K_s(\lambda) = 1/{}_2F_1(s,\nu+s;\nu+1;\lambda^2)$	15
$E_s^{(\nu)}(\lambda,\underline{\xi})$	Saw's hyperspherical t-distribution	15
$f_c(\theta)$	density for the circular cardioid distribution	17
$f_{pt}(\theta)$	density for the circular point distribution	17
$f_w(\theta)$	density for the circular wrapped normal distribution	17
$\underline{\bar{x}}_0$	direction of the sample mean	19
r_n	sample resultant length	19
ess inf	essential infimum	19
$N(\theta,x)$	number of sample points belonging to the circular arc corresponding to the interval $(x,x+\theta]$	22
$A(\theta)$	generalization of Ajne's statistic I on the circle	23
$B(\theta)$	generalization of Ajne's statistic N on the circle	23
I	Ajne's test statistic: $I = 2\pi n\, A(\pi)$	23
N	Ajne's test statistic: $N = B(\pi)$	23
H_0	null hypothesis	23
H_1	alternative hypothesis	23
$L(z)$	limiting distribution of the Kolmogorov-Smirnov statistic $D(n)$	24
$D(n)$	Kolmogorov-Smirnov statistic	24
$A_{m,n}$	m-dimensional analogue of Ajne's statistic $A(\pi)$	25
$N(\underline{\xi})$	number of sample points $\underline{x}_i$ such that $\underline{\xi}'\underline{x}_i > 0$	25

Definiendum	Definiens	Page Introduced
T_n	Beran's class of test statistics on the circle	26
$G_n^{(\nu)}$	Saw's generalization of Beran's class of test statistics to m-dimensions	26
$A_p^{(\nu)}(x)$	Genenbauer polynomial (standardization due to Saw)	26
$G_n(\underline{x})$	homogeneous polynomial of degree n in m dimensions	28
∇_m	Laplacian operator	28
$S_n(m,\underline{u})$	spherical harmonic of order n in m dimensions	28
$N(m,n)$	dimension of a basis for $\{S_n(m,\underline{u})\}$	29
$S_{n,j}(m,\underline{u})$	elements of a basis for $\{S_n(m,\underline{u})\}$	29
δ_{pq}	Kronecker's delta function	29
$P_n(m,\underline{x})$	Legendre functions of degree n in m dimensions	30
det	determinant	30
$C_n^{\nu}(x)$	Gegenbauer or ultraspherical polynomial	31
(a,m)	Appell's symbol: Let a be a complex number and let m be a positive integer. Then, $(a,m) = a(a+1)(a+2)\cdots(a+m-1)$ $(a,0) = 1$ $(a,-m) = 1/[(a-1)(a-2)\cdots(a-m)] \quad a \neq 1,2,\ldots,m.$	35
$O(\phi(x))$	order symbol: If $\lvert f(x)/\phi(x)\rvert$ is bounded as $x \to c$, then $f(x) = O(\phi(x))$ as $x \to c$.	35
D^n	differentiation operator: $D^n = d^n/dx^n \quad n = 1,2,3,\ldots$	37
$J_\nu(x)$	Bessel function of the first kind: $J_\nu(x) = (x/2)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k}}{k!\,\Gamma(\nu+k+1)}$	44
$w^{(\nu)}(x)$	$w^{(\nu)}(x) = D\,W^{(\nu)}(x) = (1-x^2)^{\nu-1/2} \,/\, B(1/2,\nu+1/2)$	47
$I^{(\nu)}(p,q,r)$	$I^{(\nu)}(p,q,r) = \int_{-1}^{1} A_p^{(\nu)}(x) A_q^{(\nu)}(x) A_r^{(\nu)}(x)\, dW^{(\nu)}(x)$	49

Definiendum	Definiens	Page Introduced
$T_p(x)$	Chebyshev polynomial of the first kind: $$T_p(x) = \frac{p}{2}\sum_{k=0}^{[p/2]} \frac{(-1)^k\,(p-k-1)!}{k!\,(p-2k)!}\,(2x)^{p-2k}$$	51
$U_p(x)$	Chebyshev polynomial of the second kind: $$U_p(x) = \sum_{k=0}^{[p/2]} \frac{(-1)^k\,(p-k)!}{k!\,(p-2k)!}\,(2x)^{p-2k}$$	51
$H_p(x)$	Hermite polynomial: $$H_p(x) = p!\sum_{k=0}^{[p/2]} \frac{(-1)^k\,(2x)^{p-2k}}{k!\,(p-2k)!}$$	52
$P_p^{(\alpha,\beta)}(x)$	Jacobi polynomial: $$P_p^{(\alpha,\beta)}(x) = 2^{-p}\sum_{k=0}^{p}\binom{p+\alpha}{k}\binom{p+\beta}{p-k}\,(x-1)^{p-k}\,(x+1)^k$$	52
$P_p(x)$	Legendre polynomial: $$P_p(x) = 2^{-p}\sum_{k=0}^{[p/2]}(-1)^k\binom{p}{k}\binom{2p-2k}{p}\,x^{p-2k}$$	53
$P_p^q(x)$	associated Legendre function of the first kind: $$P_p^q(x) = \frac{1}{\Gamma(1-q)}\left(\frac{x+1}{x-1}\right)^{q/2} F\left(-p,p+1;1-q;\frac{1-x}{2}\right),\ \left\|1-x\right\|<2$$	53
$\operatorname{sgn}(x)$	signum function: $$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$	54
$\bar{g}_p^{(\nu)}$	p^{th} coefficient in the Gegenbauer transform of a function $g(\underline{\xi}'\underline{u})$	58
$M(a,b,x)$	confluent hypergeometric function $$M(a,b,x) = {}_1F_1(a;b;x)$$	60

Definiendum	Definiens	Page Introduced
$H_m(\underline{\xi})$	m-dimensional hemisphere, rotationally symmetric rotationally symmetric about $\underline{\xi} \in S_m$	61
$I_{[A]}(x)$	indicator function: $I_{[A]}(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$	61
$X(\underline{u})$	$X(\underline{u}) = 2n^{-1/2}(N(\underline{u}) - n/2)$	62
$\delta(x)$	$\delta(x) = \operatorname{sgn}(x)$	62
$\tau(x)$	$\tau(x) = (2/\pi) \sin^{-1}x$	69
$Y_p(\underline{\xi})$	$Y_p(\underline{\xi}) = (-1)^{(p-1)/2} \int_{S_m} X(\underline{u})\, A_p^{(\nu)}(\underline{\xi}'\underline{u})\, dU^{(\nu)}(\underline{u})$	71
$h_n(r)$	density of the resultant length	76
$\phi_p(\underline{\xi}'\underline{u})$	generalized cardioid distribution	89
$V(\cdot)$	score function	89

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BIOGRAPHICAL SKETCH

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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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